Math 7: Handout 12 [2023/01/08] : Quadratic Equations

Classwork

Today we discussed how one solves quadratic equation: $ax^2+bx+c=0$. The method used is called "completing the square". Here is an example how it works:

$$x^2+6x+2=x^2+2\cdot 3x+9-7=(x+3)^2-7=(x+3+\sqrt{7})(x+3-\sqrt{7})$$

thus, $x^2+6x+2=0$ if and only if $x+3+\sqrt{7}=0$, which gives $x=-3-\sqrt{7}$, or $x+3-\sqrt{7}=0$, which gives $x=-3+\sqrt{7}$.

The same trick works in general: if $a=1$, then

$$\begin{aligned} x^2+bx+c &= x^2+2\tfrac{b}{2}x+c=\left(x^2+2\tfrac{b}{2}x+\tfrac{b^2}{2^2}\right)-\tfrac{b^2}{2^2}+c \\ &= \left(x+\tfrac{b}{2}\right)^2-\tfrac{b^2-4c}{4}=\left(x+\tfrac{b}{2}\right)^2-\tfrac{D}{4} \end{aligned} \tag{1}$$

where $D=b^2-4c$.

Thus, $x^2+bx+c=0$ is equivalent to

$$\left(x+\tfrac{b}{2}\right)^2=\tfrac{D}{4}$$

If a is not equal to 1, the answer is similar: $ax^2+bx+c=0$ is equivalent to

$$\left(x+\tfrac{b}{2a}\right)^2=\tfrac{D}{4a^2},\qquad D=b^2-4ac$$

Therefore, if $D<0$, there are no solutions; if $D\geq 0$, solutions are

$$\begin{aligned} x+\tfrac{b}{2a} &= \pm\frac{\sqrt{D}}{2a} \\ x &= \frac{-b\pm\sqrt{D}}{2a} \end{aligned} \tag{2}$$

1. Let's try solving these equations:

(a) $2x^2-11x+9=0$ (b) $6x^2+x-1=0$ (c) $x^2+12x+35=0$ (d) $x^2=1+x$

2. A rectangular 8m by 12m grass playing field is surrounded by a dirt road. How wide is the road, if its area is equal to the area of the field?

***3.** In a sequence of numbers, each is equal to the sum of the previous and $2\times$ the one before: $a_n=a_{n-1}+2a_{n-2}$. Find the general formula if $a_1=1$ and $a_2=5$.

Homework

1. Solve the following equations. Carefully write all the steps in your argument.

(a) $x^2-5x+5=0$ (b) $2x(3-x)=1$ (c) $x^3+4x^2-45x=0$ (d) $\dfrac{x}{x-2}=x-1$

2. Indian mathematicians were aware of the quadratic formula for solving quadratic equations. Can you solve the following problem by the 9th century mathematician Mahāvīra?

One-third of a herd of elephants and three times the square root of the remaining part of the herd were seen on a mountain slope; and in a lake was seen a male elephant along with three female elephants constituting the ultimate remainder. How many were the elephants here?

3. In the 12th century, Indian mathematician Bhaskara formulated the following problem. Solve it!

Out of a party of monkeys, the square of one fifth of their number diminished by three went into a cave. The one remaining monkey was climbing up a tree. What is the total number of monkeys?

4. (a) Use formula (1) to prove that for any x, $x^2+bx+c\geq -D/4$, with equality only if $x=-b/2$.
(b) Find the minimal possible value of the expression x^2+4x+2
(c) Given a number $a>0$, find the maximal possible value of $x(a-x)$ (the answer will depend on a).

5. If $x+\frac{1}{x}=7$, find $x^2+\frac{1}{x^2}$; $x^3+\frac{1}{x^3}$. Hint: examine $\left(1+\frac{1}{x}\right)^2$ and $\left(1+\frac{1}{x}\right)^3$.

***6.** Consider the sequence $x_1=1$, $x_2=\frac{x_1}{2}+\frac{1}{x_1}$, $x_3=\frac{x_2}{2}+\frac{1}{x_2}$ .... Compute the first several terms; does it seem that the sequence is increasing? decreasing? approaching some value? If so, can you guess this value? [Hint: solve equation $x=\frac{x}{2}+\frac{1}{x}$.]