Math 7: Handout 14 [2023/01/23] : General Vieta Formulas

Vieta formulas

Last time we looked at Vieta formulas for quadratic equations. That is, if x_1, x_2 are roots of quadratic polynomial $ax^2 + bx + c$, then

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1 x_2 &= \frac{c}{a} \end{aligned}$$

In addition to quadratic equations, we can also look at other types of equations:

- **Cubic equations:** These are the equations with the 3rd power terms (x^3), generally we can write them as
$$ax^3 + bx^2 + cx + d = 0$$
.
- **4-th power equations:** These are the equations with the 4th power terms (x^4), generally we can write them as
$$ax^4 + bx^3 + cx^2 + dx + e = 0$$
.
- etc...

We are not going to study cubic and other equations of higher power now. It is suffice to say that there are formulas for solving cubic equation (Cardano's Formula), and even formulas for solving equations of the 4th power — but they are rarely used, and are pretty large. It can also be proven that equations of 5th power of higher do not have a formula, and it is impossible to find a formula like that.

Interestingly, Vieta formulas can be generalized for an equation of any higher power. Similarly to what we did for quadratic equations, if the equation of degree n

$$p(x) = ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots + w$$

has n roots $x_1, x_2, \ldots, x_n$ then one can write it as

$$p(x) = a(x - x_1)\ldots(x - x_n)$$

. Expanding the right hand side we obtain Vieta formulas:

$$\begin{gathered} x_1 + x_2 + \cdots + x_n = -\tfrac{b}{a} \\ x_1x_2 + x_1x_3 + \cdots + x_2x_3 + \cdots = \tfrac{c}{a} \\ x_1x_2x_3 + x_1x_2x_4 + \cdots + x_2x_3x_4 + \cdots = -\tfrac{d}{a} \\ \cdots \\ x_1x_2\ldots x_n = (-)\tfrac{w}{a} \end{gathered}$$

That is, the sum of all roots is $-\frac{b}{a}$, the sum of all possible pairwise products of roots is $\frac{c}{a}$, etc., until we get to the product of all roots being equal to $\frac{w}{a}$ with an appropriate sign – notice, the signs they alternate.

Classwork

1. Without solving the equation $x^3 - 5x + 6x - 1 = 0$, find

(a) $x_1 + x_2 + x_3$ (b) $x_1^2 + x_2^2 + x_3^2$ (c) $x_1^3 + x_2^3 + x_3^3$

2. Without solving the equation $x^2 - 12x + 19 = 0$ find the value of the expression $x_1(1 - x_1) + x_2(1 - x_2)$.

3. Solve the equation $(x^2 + 2)^2 = 6x^2 + 4$. [**Hint**: Of course, you can just use the $(a+b)^2$ formula. Alternatively, one of the ways to solve it is to assume that $t = x^2 + 2$. Then the equation can be rewritten as a quadratic equation with t as a variable.]

4. Find the minimal value of these polynomials:

(a) $x^2 + 6x + 4$ (b) $x^2 - 5x + 8$ (c) $x^2 + 3x + 10$ (d) $x^2 - 4x + 12$

HOMEWORK

1. Find the roots of the equation $4x^2 - 2x - 1 = 0$ **WITHOUT** using the formula for roots of quadratic equation. That is, complete the square and use the difference of squares formula to factorize the polynomial.
2. What is the sum of the roots $(x_1 + x_2 + x_3)$ of the equation $x^3 - 6x^2 + 11x - 6 = 0$? What is the product of those roots $(x_1x_2x_3)$? Can you guess the roots?
3. Without solving the equation $3x^2 - 5x + 1 = 0$ find the arithmetic mean of its roots (that is $\frac{x_1+x_2}{2}$) and their geometric mean (that is $\sqrt{x_1x_2}$).
4. Find all numbers a such that sum of squares of the roots $(x_1^2 + x_2^2)$ of the equation $x^2 - ax + a + 7 = 0$ is equal to 10.
5. If x_1, x_2 are solution fo the quadratic equation $x_2 - 5x + a^2 - 2a + 1$, where a is some number. Find the value of a so that the product of solutions of the equation is minimal.
6. Solve the equation $x^4 - x^2 - 2 = 0$.
7. For a given a, what is the minimum of the value of the expression $x^2 - ax + 1$? At what x that expression has a minimum value?
8. In a right angle triangle, one leg is 4 inches shorter than the hypothenuse, and the other leg is 2 inches shorter than the hypothenuse. Find the length of the hypothenuse.