Math 7: Handout 16 [2023/02/05] : More Inequalities. Snake Method

POLYNOMIAL INEQUALITIES

Today we continue studying inequalities. First recall polynomial inequalities: they have terms like x^2, x^3, etc. i The general rule for solving polynomial inequalities is as follows:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$ (for polynomial of degree more than 2, you would have more factors).
- Roots $x_1, x_2, \ldots$ divide the real line into intervals; define the sign of each factor and the product on each of the sign intervals.
- If the inequality has $\geq$ or $\leq$ signs you should also include the roots themselves into the intervals.

Example 1. $x^2 + x - 2 > 0$.

Solution. We find roots of the equation $x^2 + x - 2 = 0$ and obtain $x = -2, 1$. The inequality becomes $(x + 2)(x - 1) > 0$ and roots -2, 1 divide the real line into three intervals $(-\infty, -2), (-2, 1), (1, +\infty)$. It is easy to see that the polynomial $x^2 + x - 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then $x < -2$ or $x > 1$. We sometimes, write this also as $x \in (-\infty, -2) \cup (1, +\infty)$. (sign $\cup$ means “or”).

Practice exercises

1. $-x^2 - x + 2 \geq 0$; **2.** $x^2 + x + 2 \geq 0$; **3.** $x^2 + x + 2 < 0$; **4.** $x^2 - 2x + 1 > 0$.

The same method can be used to solve any polynomial inequality, for example $x^n + bx^{n-1} + \cdots \geq 0$, where n is greater than 2 — but we need to know the way to either find the roots of the corresponding equation or to have factorization given to us.

Example 2. Solve the inequality $(x + 1)(x - 2)^2(x - 4)^3 \geq 0$.

Solution. Notice that if we solve the corresponding equation $(x + 1)(x - 2)^2(x - 4)^3 = 0$, we get $x = -1, 2, 4$. Therefore, we need to consider the following 4 intervals: $(-\infty; -1), (-1; 2), (2; 4), (4; \infty)$.

- Notice that in the 1st interval, the expression $(x + 1)(x - 2)^2(x - 4)^3$ is positive, and therefore satisfies the inequality.
- Then, as x “crosses” point 1, the expression changes its sign to ‘$-$’, and therefore the interval $(-1; 2)$ does not satisfy the inequality.
- Now, crossing point 2 again won’t change the sign of the expression, because $(x - 2)^2$ is always positive. Therefore, the interval $(2; 4)$ also doesn’t satisfy the inequality.
- Finally, crossing point 4, the expression changes its sign to ‘$+$’, and therefore the interval $(4; \infty)$ satisfies the inequality. So, the answer to the inequality is:

$$x \in (-\infty; -1] \cup 2 \cup [4; \infty)$$

The method used to solve this problem is called a **snake method**.

Example 3. Solve the inequality $\frac{(x+1)(x-2)^2}{(x-4)^3} \geq 0$.

Solution. Note that the factors in this inequality are exactly the same as in the previous example, so the solution will be the same with the small (but important) exception: the denominator cannot be equal to 0, and therefore, x cannot be equal to 4 — notice the round instead of square bracket in the answer!

$$x \in (-\infty; -1] \cup 2 \cup (4; \infty)$$

INEQUALITIES WITH ABSOLUTE VALUE

When you have an inequality with absolute value, you will have to consider various cases: when the expression under absolute value is positive and when the expression under the absolute value is negative, and use the definition of the absolute value:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } -x \geq 0 \end{cases}$$

Example 4. Solve inequality $|x-4| < 7$.

Solution. Solution: Again, as before, we need to consider two cases, the one when $x-4 \geq 0$ and the one when $x-4 < 0$.
Case 1. $x-4 \geq 0$ means that $x \geq 4$. Now, since $x-4 \geq 0$, we have $|x-4| = x-4$, and the inequality can be rewritten as

$$x - 4 < 7$$

Solving this inequality gives us $x < 11$. But remember, x must be greater than or equal to 4! So, combining both things together, we get $4 \leq x < 11$, or $x \in [4; 11)$.
Case 2. $x-4 < 0$ means that $x < 4$. Now, since $x-4 < 0$, we have $|x-4| = -(x-4) = 4-x$, and the inequality can be rewritten as

$$4 - x < 7$$

Solving this inequality gives us $x > -3$. But remember, x must also be less than 4! So, combining both things together, we get $-3 < x \leq 4$.

Combining Cases 1 and 2 together, we get the final solution to the inequality: $-3 < x < 11$ or

$$x \in (-3, 11)$$

HOMEWORK

1. Solve the following equations.

(a) $|x-3| = 5$ (b) $|2x-1| = 7$ (c) $|x^2-5| = 4$

2. Solve the following equations.

(a) $\dfrac{(x+1)}{(x-1)} = 3$ (b) $\dfrac{(x^2-9)}{(x+1)} = (x+3)$ (c) $x - \dfrac{3}{x} = \dfrac{5}{x} - x$

3. Solve the following inequalities, show solution on the real line, write the answer in the interval notation.

(a) $|x-2| > 3$ (b) $|x-1| > x+3$ (c) $\dfrac{(x-2)}{(x+3)} \leq 3$

4. Solve the following quadratic inequalities:

(a) $x^2+2x-3 > 0$, (b) $x^2+2x+3 < 0$ (c) $-x^2+6x-9 > 0$ (d) $3x^2+x-1 < 0$

5. Solve the following inequalities, using the snake method. Show solution on the real line. Write the answer in the interval notation.

(a) $(x+3)(x-2)^2 \leq 0$ (b) $x(x-1)(x+2) \geq 0$ (c) $\frac{x^2(x+1)^5(x+2)^3}{x-1} > 0$