Math 7: Handout 18 [2023/02/26] :
Coordinate Geometry 1: Review. Lines and Circles. Basic Transformations

1. Coordinate Geometry: Introduction

In this section of the course we are going to study coordinate geometry. The basic notion is the **coordinate plane** – a plane with a given fixed point, called the **origin**, as well as two perpendicular lines – **axes**, called the x**-axis** and the y**-axis**. x-axis is usually drawn horizontally, and y-axis — vertically. These two axes have a **scale** – "distance" from the origin.

The scales on the axes allow us to describe any point on the plane by its **coordinates**. To find coordinates of a point P, draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates x_0 and y_0. Then the point P has x-coordinate x_0, and y-coordinate y_0, and the notation for that is: $P(x_0, y_0)$.

The **midpoint** M of a segment AB with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ has coordinates:

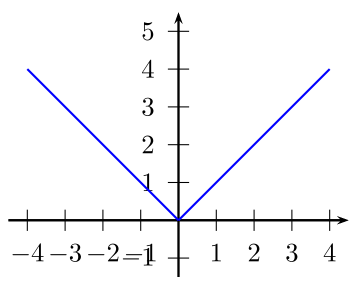
$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

2. Graphs of functions

Given some relation which involves variables x, y (such as $x+2y = 0$ or $y = x^2+1$), we can plot on the coordinate plane all points $M(x, y)$ whose coordinates satisfy this equation. Of course, there will be infinitely many such points; however, they usually fill some smooth line or curve. This curve is called the **graph** of the given relation.

In general, the relation between x and y could be more complicated and could be given by some formula of the form $y = f(x)$, where f is some function of x (i.e., some formula which contains x). Then the set of all points whose coordinates satisfy this relation is called the **graph** of f.

The figure below shows a graph of a function $y = |x|$.

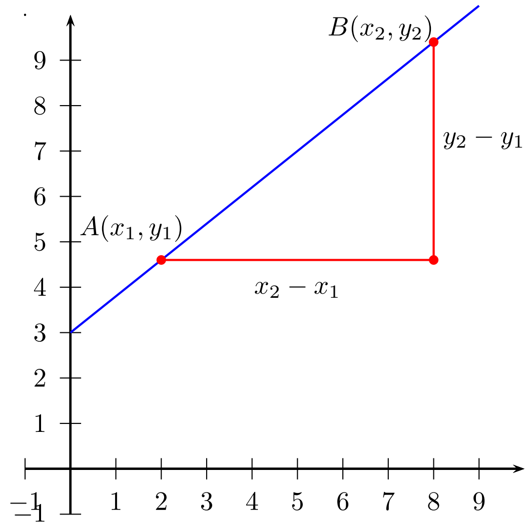


3. Lines

Every relation (**equation**) of the form:

$$y = mx + b$$

where m, b are some numbers, defines a straight line. The slope of this line is determined by m: as you move along the line, y changes m times as fast as x, so if you increase x by 1, then y will increase by m:

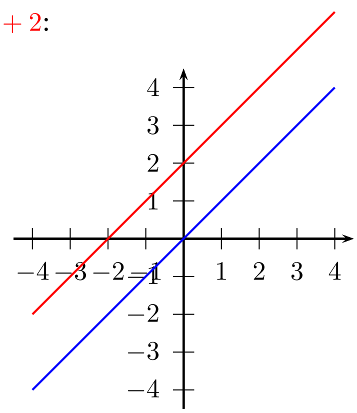


In other words, given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ **slope** can be computed by dividing change of y: $y_2 - y_1$ by the change of x: $x_2 - x_1$:

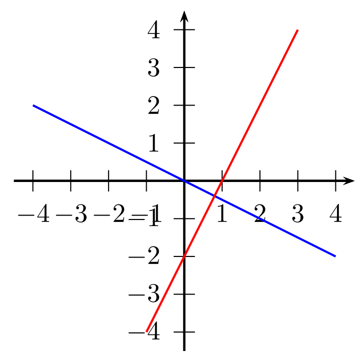
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Two lines are **parallel** if and only if they have the **same slope**.
- Two lines with slopes m and n are **perpendicular** if and only if $m = -\frac{1}{n}$.

$y = x$; $y = x + 2$:



$y = -\frac{1}{2}x$; $y = x - 2$:



In the equation $y = mx + b$, b is a y**-intercept**, and determines where the line intersects the vertical axis (y-axis).

The equation of the **vertical** line is $x = k$, and the equation of the **horizontal** line is $y = k$.

If the line is vertical, the slope is undefined (infinity), and the intercept is also undefined (zero or infinity).

Exercise 1 Write equation for a line *parallel* to $y = 3x - 1$ and passing through point $(3, 4)$.

Exercise 2 Write equation for a line *perpendicular* to $y = -\frac{3}{2}x + 2$ and passing through point $(6, 2)$.

4. Distance Between Points. Circles

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the following formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This formula is a straightforward consequence of the Pythagoras' Theorem.

The equation of the circle with the center $M(x_0, y_0)$ and radius r is

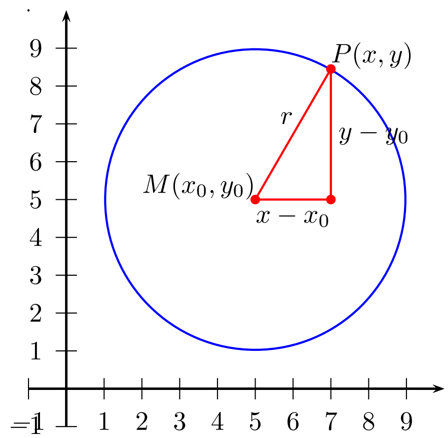
$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

This equation means, that points (x, y) should be at distance r from the given point $M(x_0, y_0)$.

Equation for a circle can also be written as

$$y = \pm\sqrt{r^2 - (x - x_0)^2} + y_0 \,,$$

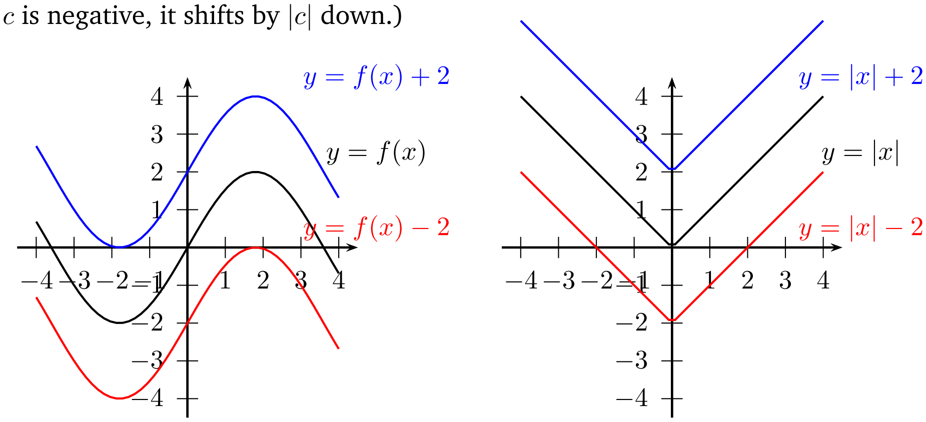
and the $(\pm)$ signs correspond to the upper and the lower arcs (half-circles).



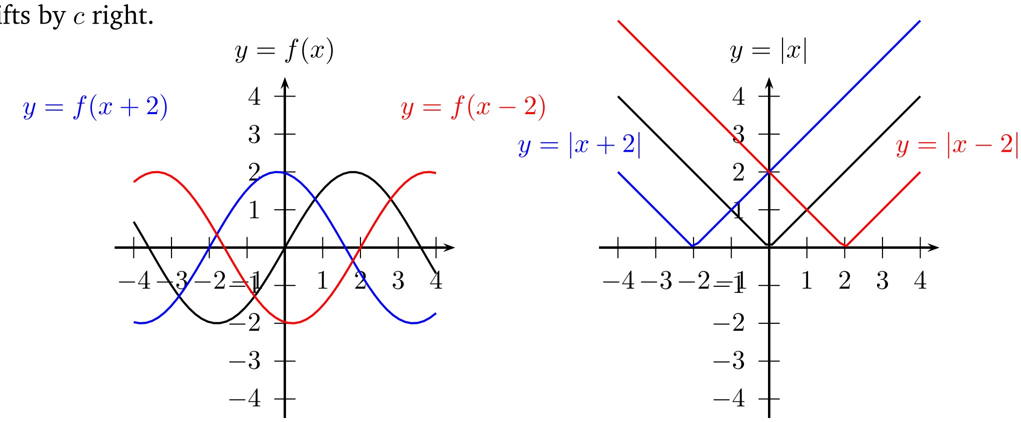
5. Trasnformations

Having learned a number of basic graphs, we can produce new graphs, by doing certain transformations of the equations. Here are two of them.

Vertical translations: Adding constant c to the right-hand side of equation shifts the graph by c units up (if c is positive; if c is negative, it shifts by $|c|$ down.)



Horizontal translations: Adding constant c to x shifts the graph by c units left if c is positive; if c is negative, it shifts by c right.



A way to memorize these transformations is to think about "new" x and y compared to "old" x and y:

$$\begin{cases} x_{\text{new}} = x_{\text{old}} + [\text{ shift-right}]\,, \\ y_{\text{new}} = y_{\text{old}} + [\text{ shift-up}]\,, \end{cases} \qquad \text{and if } y_{\text{old}} = f(x_{\text{old}})\,, \quad \text{then } y_{\text{new}} - [\text{ shift-up}] = f(x_{\text{new}} - [\text{shift-right}])$$

Homework

Please use graph ruled paper for this homework - it makes everything much easier!

1. A point B is 5 units above and 2 units to the left of point $A(7,5)$. What are the coordinates of point B?

2. Find the coordinates of the midpoint of the segment AB, where $A = (3,11)$, $B = (7,5)$.

3. Draw points $A(4,1)$, $B(3,5)$, $C(-1,4)$. If you did everything correctly, you will get 3 vertices of a square. What are coordinates of the fourth vertex? What is the area of this square?

4. 3 points $(0,0)$, $(1,3)$, $(5,-2)$ are the three vertices of a parallelogram. What are the coordinates of the remaining vertex?

5. What is the slope of a line whose equation is $y = 2x$? What is the slope of a line perpendicular to it?

6. In this problem you will find equations that describe some lines.
- (a) What is the equation whose graph is the y-axis?
- (b) What is the equation of a line whose points all lie 5 units above the x-axis?
- (c) Is the graph of $y = x$ a line? Draw it.
- (d) Find the equation of a line that contains the points $(1,-1)$, $(2,-2)$, and $(3,-3)$.

7. For each of the equations below, draw the graph, then draw the line perpendicular to it and going through the point $(0,0)$, and then write the equation of the perpendicular line

(a) $y = 2x$ (b) $y = 3x$ (c) $y = -x$ (d) $y = -\frac{1}{2}x$

8.
- (a) Find the equation of the line through $(1,1)$ with slope 2.
- (b) Find the equation of the line through points $(1,1)$ and $(3,7)$. [Hint: what is the slope?]
- (c) Find k if $(1,9)$ is on the graph of $y - 2x = k$. Sketch the graph.
- (d) Find k if $(1,k)$ is on the graph of $5x + 4y - 1 = 0$. Sketch the graph.

9. Find the intersection point of a line $y = x - 3$ and a line $y = -2x + 6$. Sketch the graphs of these lines.

10. Sketch graphs of the following functions:

(a) $y = |x| + 1$ (b) $y = |x + 1|$ (c) $y = |x - 5| + 1$