Homework 9: Binomial coefficients and Pascal's triangle

HW9 is Due December 5; submit it to Google classroom 15 minutes before the class time.

1. Pascal's triangle and the binomial coefficients.

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&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

Every entry in it is obtained as the sum of two entries above it. The k-th entry in the n-th line is denoted by $\binom{n}{k}$, or by nC_k . Note that both n and k are counted from 0, not from 1:
for example, $\binom{6}{2} = 15, \binom{3}{0} = 1$. Now let us think about other applications of these numbers.

2. Example applications of the binomial coefficients

a) Chessboard paths

In the previous handout, we saw that these numbers appear in a problem about counting paths from the lower left corner of the board to the upper right corner. We observed the following:

$\binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k units to the right.

For example, the number of paths that go to the upper right corner of a 6 × 6 board is equal to $\binom{10}{5}$, as *each* path must have 5 steps to the right and 5 steps up, taken in any order. This means that we have a total of n = 10 steps made of k = 5 steps up and the rest, 10 - 5 = 5 steps to the right. Or, from 10 possible moves, we pick 5 to be up.

b) Words with Rs and Us or 1s and 0s

Each of these paths on the board going up and to the right can be written as a sequence of steps; let R be a step to the right, and U a step up. For example, a possible path is: RRRRRUUUUU will go five steps to the right and five steps up, eventually ending in the upper right corner of a 6 × 6 board. Another possible sequence is RRUUURRRUU. There is a correspondence between paths of length **n** and strings of length **n** consisting of ***Rs*** and ***Us*** only. Now let us switch Rs to 0s and Us to 1s. We already know that $\binom{n}{k}$ = is the number of paths going k units up and n − k units to the right. This corresponds to words of length n with k Us and n − k Rs, which is the same as the number of strings of length n with k 1s and n − k Us. We have the following result:

$\binom{n}{k}$ = the number of words with length n that can be written using k ones and n − k zeroes or k Us and n-k Rs.

c) Combinations: *the choice of **k** things from a set of **n** things without repetition ("replacement") and where the order does not matter.*

Consider now a set of n elements; let's number them from 1 to n. Then, for each string of length n with 0s and 1s, we can select those elements that correspond to 1s and omit those elements that correspond to 0s. In this way, we will get <u>a subset of size k</u>. This is another property of the binomial coefficients:

$\binom{n}{k}$ = the number of ways to choose k items out of n (order doesn't matter)

3. Summary: binomial coefficients represent

$nC_k = \binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k to the right
= the number of words that can be written using k ones and n − k zeroes
= the number of ways to choose k items out of n (order doesn't matter)

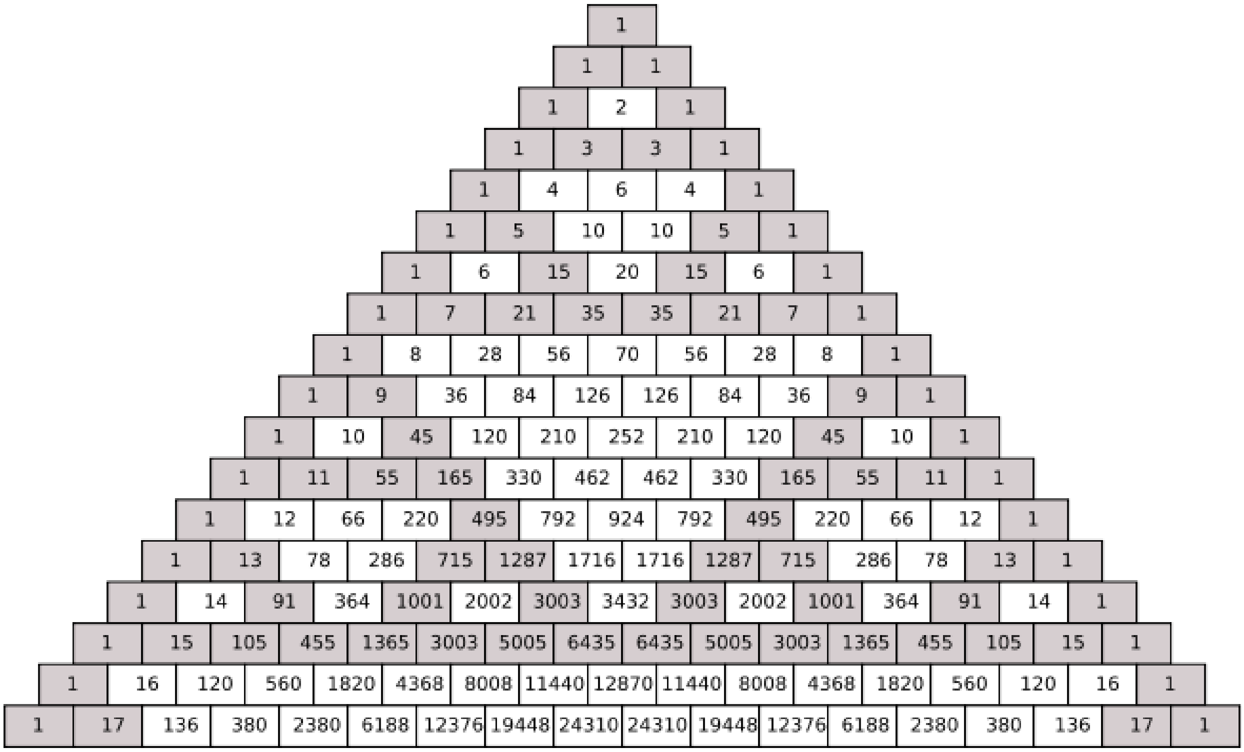
Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

Note: In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use the Pascal's triangle to find the value of $\binom{n}{k}$, where k is the k-th element in the n-th row of the Pascal triangle, counting from 0.

1. How many "words" of length 5 one can write using only letters U and R, namely 3 U's and 2 R's? What if you have 5 U's and 3 R's? [Hint: each such "word" can describe a path on the chessboard, U for up and R for right. . .]
2. How many sequences of 0 and 1 of length 10 are there? sequences of length 10 containing exactly 4 ones? exactly 6 ones?
3. If we toss a coin 10 times, what is the probability that all will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?
4. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub — and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 10 steps, he will move:
   a) 10 steps north
   b) 10 steps south
   c) 4 steps north
   d) will come back to the starting position
5. If you have a group of 4 people, and you need to choose one to go to a competition, how many ways are there to do it? if you need to choose 2? if you need to choose 3?
6. How many ways are there to select 5 students from a group of 12?
7. In a meeting of 25 people, all must shake hands with each other. How many handshakes are there altogether?
8. (a) An artist has 12 paintings. He needs to choose 4 paintings to include in an art show. How many ways are there of doing this?

   (b) The same artist now needs to choose 4 paintings to include in a catalog. How many ways are there to do this? (In the catalog, unlike the show, the order matters).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