Homework 8: Introduction to Combinatorics – Pascal's triangle

HW8 is Due November 30; submit it to Google classroom 15 minutes before the class time.

1. Fundamental Principle of Counting (Multiplication rule).

Combinatorics = counting

If the first task can be performed in **m** ways, and for each of these a second task can be performed in **n** ways, and for each combination, a third task can be performed in **k** ways, etc. then this entire sequence of tasks can be performed in **m· n · k** . . . ways.

2. Permutations: *the choice of **k** things from a set of **n** things without repetition("replacement") and where the order matters.*

a) Picking first, second, and third place winners from a group. No replacement – cannot pick the same person twice, order matters – Dave, Emma, Page is a different group than Emma, Dave, Page.
If a group has **n** members, then picking 3 can be done in

$$n\,P_3 = n(n-1)(n-2) \quad \text{ways}$$

b) Permutations of **n** things: The number of permutations of all n different things: n!

- Arranging/ordering all n members of a group can be done in n! ways.
- Listing the favorite desserts in the order of choices: if there are n desserts in total, there are n! ways to arrange them in the order of preference.

3. Combinations: *the choice of **k** things from a set of **n** things without repetition ("replacement") and where **the order does not matter**.*

Combinations are harder to count; we will talk about them later and write the formulas!

- Picking three team members from a group (it doesn't matter who is chosen first, or second, or third).
- Picking two desserts from a tray (the order in which you eat them doesn't matter!).

4. PASCAL's TRIANGLE

How many ways are there to go from the bottom left corner of the chessboard to the upper right, moving always only to the right and up?

1	6	21	56	126	252
1	5	15	35	70	126
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

To make out life easier, we will refer to cells by two numbers **(m, n)**: m is the number of the column (counting from left), and n is the number of the row (counting from the bottom). We can solve this problem iteratively. There is only one path to any of the cells in the lowest row or the left column. Let's put ones there. Now, let's think about other cells. To get to cell (2, 2), we can first get to cell (2, 1) (there's only 1 way to get there), and then do a step up; or first get to cell (1, 2) (there's again only 1 way to get there), and then do a step right. That means, there are 1 + 1 = 2 ways to get to cell (2, 2).

Now, let's think about cell (3, 2). Again, we have two choices: first, we can get there from cell (2, 2) by doing a step right, or second, we can get there from cell (3, 1) by doing a step up. It means the total number of paths to get to cell (3, 2) will be equal to the total number of ways to get to (2, 2) plus the total number of ways to get to (3, 1): 2 + 1 = 3. Keeping this process going, we can notice that the number of paths to cell (m, n) is equal to the number of paths to cell (m − 1, n) plus the number of paths to cell (m, n − 1). This way we can fill out the entire table (finish it as a part of your homework).

These numbers are called the binomial coefficients. They are usually written in a slightly different way:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
. . .

This triangle is called **Pascal's triangle**. Every entry in it is obtained as the sum of two entries above it. The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by nC_k . Note that both n and k are counted from 0, not from 1:
for example, $\binom{2}{1} = 2$, $\binom{3}{0} = 1$.

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer and some justification for why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Dinner in a restaurant consists of 3 courses: appetizer, main course, and dessert. There are 5 possible appetizers, 6 main courses, and 3 desserts. How many possible dinners are there?
2. How many ways are there to seat 5 students in a class that has 5 desks? if there are 10 desks?
3. How many ways are there to select first, second, and third prize winners if there are 14 athletes in a competition?
4. How many ways are there to put 8 rooks on a chessboard so that no one attacks the others?
5. A dressmaker has two display windows. The left display is for evening dresses and the one in the right window is for regular day dresses. Assuming she can put 10 evening dresses in any order, and separately, 5 regular dresses in any order, how many total possibilities for arranging the two display windows are there?
6. The guidelines at a certain college specify that for the introductory English class, the professor may choose one of 3 specified novels, and choose two from a list of 5 specified plays. Thus, the reading list for this introductory class must have one novel and two plays. How many different reading lists could a professor create within these parameters?
7. Finish the chessboard problem (for 8×8-board: how many ways are there to go from the lower left corner to the upper right corner?
8. Which of the numbers in the Pascal's triangle are even? Can you *guess the pattern*, and then carefully explain why it works? (no formulas)
9. What is the sum of all entries in the n-th row of the Pascal's triangle? Try computing the first several answers and then guess the general formula.

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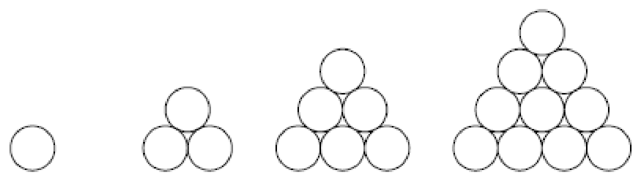
10. What is the alternating sum of all the numbers in n-th row of the Pascal's triangle, i.e.

$$1 + \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \dots$$

Try computing the first several sums and then guess the general formula.

11. **(Optional)** Let us draw a figure consisting of n rows of circles as shown in the figure below (for n = 1, 2, 3, and 4):



Let T_n be the number of circles in n-th figure (for example, $T_1 = 1, T_2 = 3, T_3 = 6\dots$). These numbers are sometimes called triangular numbers.

(a) What is the difference $T_{n+1} - T_n$? Try for a few n as 1, 2, 3, 4, 5, …

(b) Show that the numbers T_n appear in the Pascal triangle as shown below, that is $T_n = \binom{n+1}{2}$. Again, try n = 1, 2, 3, 4 …

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & \boxed{1} & & \\ & 1 & & 3 & & \boxed{3} & & 1 & \\ 1 & & 4 & & \boxed{6} & & 4 & & 1 \\ & & & & \cdots & & & & \end{array}$$