

## Homework 16: Generalized Vieta's formulas

HW16 is Due February 8; submit it to Google classroom 15 minutes before the class time.

### 1. Solving the complete quadratic equation

- By completing the square

"Completing the square" works by using the formulas for fast multiplication  $(a \pm b)^2 = a^2 \pm 2ab + b^2$  (\*)

Here is an example of how to rewrite the standard form of an equation to factorized form by completing the square:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x + 3)^2 - 7 = (x + 3)^2 - (\sqrt{7})^2 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

Thus,  $x^2 + 6x + 2 = 0$  if and only if  $(x + 3 + \sqrt{7}) = 0$ , which gives  $x = -3 - \sqrt{7}$ , or if  $(x + 3 - \sqrt{7}) = 0$ , which gives  $x = -3 + \sqrt{7}$ .

- By using the quadratic formula. **Steps:** for the equation in the standard form  $ax^2 + bx + c = 0$

List coefficients:  $a =$  ,  $b =$  ,  $c =$

Find the determinant D:  $D = b^2 - 4ac$

Check the number of roots (solutions): The determinant, D, determines the number of solutions.

If  $D < 0$ , there are no real solutions; if  $D = 0$ , there is one solution, if  $D > 0$ , there are two solutions.

Find the solutions:  $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$

### 2. Generalized Vieta's formulas

Last time we looked at Vieta's formulas for quadratic equations. That is, if  $x_1, x_2$  are roots of a quadratic polynomial (the quadratic equation written in a standard form)  $ax^2 + bx + c = 0$

$$x_1 x_2 = \frac{c}{a} \quad \text{and} \quad x_1 + x_2 = -\frac{b}{a}$$

In addition to quadratic equations, we can also look at other types of equations:

- Cubic equations: These are the equations with the 3rd power terms ( $x^3$ ), generally written as

$$ax^3 + bx^2 + cx + d = 0$$

- 4-th power equations: These are the equations with the 4th power terms ( $x^4$ ), generally written as

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

- other equation of higher power of  $x \dots$

We are not going to study cubic and other equations of higher power now. It is sufficient to say that there are formulas for solving cubic equations (Cardanos Formula), and even formulas for solving equations of the 4th power - but are rarely used and are pretty long. It can also be proven that equations of 5th or a higher power do not have a formula, and it is impossible to find a formula for them.

Interestingly, Vieta formulas can be generalized for an equation of any higher power and they work for real and complex solutions. Similarly to what we did for quadratic equations, if an equation of degree  $n$

$$p(x) = ax^n + bx^{(n-1)} + cx^{(n-2)} + dx^{(n-3)} + \dots + w = 0$$

has  $n$  roots  $x_1, x_2, \dots, x_n$ , then one can write it as:  $p(x) = a(x - x_1) \dots (x - x_n) = 0$

Expanding the right-hand side, we obtain Vieta formulas:

$$\begin{aligned} x_1 + x_2 + \dots + x_n &= -\frac{b}{a} \\ x_1 x_2 + x_1 x_3 + \dots + x_2 x_3 + \dots &= \frac{c}{a} \\ x_1 x_2 x_3 + x_1 x_2 x_4 + \dots + x_2 x_3 x_4 + \dots &= -\frac{d}{a} \\ &\dots \\ x_1 x_2 \dots x_n &= (-)^n \frac{w}{a} \end{aligned}$$

That is, in the generalized Vieta's formulas the sum of all roots is  $-\frac{b}{a}$ , the sum of all possible pairwise products of roots is  $\frac{c}{a}$ , etc., until we get to the product of all roots being equal to  $\frac{w}{a}$  with an appropriate sign. Notice, the signs alternate.

3. **Formulas for fast multiplication**  $a^2 - b^2 = (a - b)(a + b)$ ,  $(a \pm b)^2 = a^2 \pm 2ab + b^2$ .

### Homework problems

**Instructions:** Please always write solutions on a **separate sheet of paper**. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Find the roots of the equation  $4x^2 - 2x - 1 = 0$  without using the quadratic equation formula. Here, complete the square and use the difference of squares formula to factorize the polynomial.
2. What is the sum of the roots of the equation  $x^3 - 6x^2 + 11x - 6 = 0$ ? What is the product of those roots? Could you guess the roots?
3. Without solving the equation  $3x^2 - 5x + 1 = 0$ , find the arithmetic mean of its roots (that is  $\frac{x_1 + x_2}{2}$ ) and their geometric mean (that is  $\sqrt{(x_1 x_2)}$ ).
4. Without solving the equation  $x^2 - 12x + 19 = 0$  find the value of the following expression:  
$$x_1(1 - x_1) + x_2(1 - x_2).$$
5. **(\* Optional)** Find all numbers  $p$  such that the sum of squares of the roots ( $x_1^2 + x_2^2$ ) of the equation  $x^2 - px + p + 7 = 0$  is equal to 10.
6. **(\* Optional)** If  $x_1, x_2$  are solutions for the quadratic equation  $x^2 - 5x + p^2 - 2p + 1 = 0$ , where  $p$  is some number, find the value of  $p$  so that the product of solutions of the equation is minimal.
7. Solve the equation  $x^4 - x^2 - 2 = 0$  by substitution. [Hint: set the  $x^2 = u$ .]
8. Solve the equation  $(x^2 + 2)^2 = 6x^2 + 4$ . [Hint: Of course, you can just use the  $(a + b)^2$  formula. Alternatively, one of the ways to solve it is to assume that  $t = x^2 + 2$ . Then the equation can be rewritten as a quadratic equation with  $t$  as a variable.]
9. In a right-angle triangle, one leg is 4 inches shorter than the hypotenuse, and the other leg is 2 inches shorter than the hypotenuse. Find the length of the hypotenuse.
10. **(\* Optional)** Find all numbers  $p$  and  $q$  such that the equation  $x^2 + px + q = 0$  has solutions  $p$  and  $q$ . {Hint: use Vieta's formulas}