MATH 8: HOMEWORK 01
REVIEW I SOLUTIONS

1. Open parentheses and expand the following expressions

(a) $(a+b)^2 = a^2 + 2ab + b^2$

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2$$

(b) $(a-b)^2 = a^2 - 2ab + b^2$

$$(a-b)^2 = (a-b)(a-b) = a(a-b) - b(a-b) = a^2 - ab - ba + b^2$$

(c) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$$(a-b)(a-b)^2 = (a-b)(a^2-2ab+b^2) = a(a^2-2ab+b^2) - b(a^2-2ab+b^2)) = a^3 - 2a^2b + ab^2 - ba^2 + 2ab^2 - b^3$$

2. Factor the following expressions:

(a) $a^2 - b^2 = (a-b)(a+b)$

Check by multiplying the LHS $(a-b)(a+b) = a(a+b) - b(a+b) = a^2 + ab - ba - b^2 = a^2 - b^2$

(b) $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

Multiply the LHS $(a-b)(a^2+ab+b^2) = a(a^2+ab+b^2) - b(a^2+ab+b^2) = a^3 + \underline{a^2b} + ab^2 - \underline{ba^2} - bab - b^3$

(c) $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

Multiply the LHS $(a+b)(a^2-ab+b^2) = a(a^2-ab+b^2) + b(a^2-ab+b^2) = a^3 - \underline{a^2b} + ab^2 + \underline{ba^2} - bab + b^3$

3. Expand as sums of powers of x:

$$(2x+1)^2(2-3x)$$

First write the formula for the square

$$(2x+1)^2(2-3x) = (4x^2+4x+1)(2-3x)$$

Apply the distributivity rules and open up the first paranthesis

$$= 4x^2(2-3x) + 4x(2-3x) + 1(2-3x) = \underline{8x^2} - 12x^3 + 8x - \underline{12x^2} + 2 - 3x$$

$$= -12x^3 - 4x^2 + 5x + 2$$

Placement Test 5 If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2}$, $x^3 + \frac{1}{x^3}$
If $x + 1/x = 7$ square both sides and

$$49 = (x + \frac{1}{x})^2 = x^2 + 2x\frac{1}{x} + \frac{1}{x^2} = x^2 + \frac{1}{x^2}$$

Now raise to the cube both sides

$$343 = (x + \frac{1}{x})^3 = x^3 + 3x^2\frac{1}{x}3x\frac{1}{x^2} + \frac{1}{x^3} = 3(x + \frac{1}{x}) + x^3 + \frac{1}{x^3} = 3 \cdot 7 + x^3 + \frac{1}{x^3}$$

4. A group of 19 people want to select a chairperson and two associates. How many ways there are for them to do so?

Step-by-step Solution:

First, the total number of ways to choose from 19 people a person to be the chairperson is 19
Second, the total number of ways we can place the two people from the remaining 18 onto two associates is 2-combinations out of 18, 18-choose-2: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = 9 \cdot 17$=153.

Why ? To choose the first associate we have 18 possibilities, to choose the second we have 17, so $18 \cdot 17$, but the associates positions are undistinguishable so we overcounted $\frac{18 \cdot 17}{2}$ Together with the chairperson is $\frac{19 \cdot 18 \cdot 17}{2} = 2907$

6. Consider the following quadratic equation:

$$x^2 - 5x - 14 = 0$$

- What is the discriminant of this equation?
- Solve the equation and sketch a graph of this quadratic polynomial

Solution:
The determinant is:

$$D = b^2 - 4c = (-5)^2 - 4 \cdot 1 \cdot (-14) = 25 + 56 = 81 = 9^2 => \sqrt{D} = 9$$

Solve the quadratic equation. To solve the quadratic we can either complete it to a square or use the determinant formula:
"Completing the Square" $ax^2 + bx + c = x^2 - 5x - 14 = 0$, with $a = 1$, so we use a trick $2\frac{b}{2} = b$ to create a perfect square

$$\begin{aligned} x^2 - 5x - 14 &= x^2 - 2 \cdot \tfrac{5}{2}x + \tfrac{5}{2}^2 - \tfrac{5}{2}^2 - 14 = \\ &= x^2 + 2 \cdot \tfrac{3}{2}x - \tfrac{25}{4} - \tfrac{25}{4} - 14 = (x - \tfrac{5}{2})^2 - \tfrac{25 + 4 \cdot 14}{4} \\ &= (x - \tfrac{5}{2})^2 - \tfrac{81}{4} = (x - \tfrac{5}{2})^2 - (\tfrac{9}{2})^2 \end{aligned}$$

This is a difference of two squares $u^2 - v^2$, with $u = (x - \frac{5}{2})$ and $v = \frac{9}{2}$. Then we can rewrite:

$$x^2 - 5x - 14 = (x - \tfrac{5}{2} - \tfrac{9}{2})((x - \tfrac{5}{2} + \tfrac{9}{2}) = (x - 7)(x + 2)$$

thus, $x^2 - 5x - 14 = 0$ iff $x = 7$, or $x = -2$.

We can also use directly **Determinant formula**
If we have $ax^2 + bx + c = 0$ equivalent to

$$\left(x + \tfrac{b}{2a}\right)^2 = \tfrac{D}{4a^2}, \qquad D = b^2 - 4ac$$

If $D < 0$, there are no solutions in real numbers; if $D \geq 0$, the solutions are

(1) $$x + \tfrac{b}{2a} = \pm\frac{\sqrt{D}}{2a}, \text{ or equivalently } x = \frac{-b \pm \sqrt{D}}{2a}$$

Graph the Parabola
End behavior of this quadratic: It stays positive both as x becomes larger in the positive direction, $x \longrightarrow \infty$ or if x becomes larger in the negative direction, $x \longrightarrow -\infty$ So the parabola is "holding water" Every parabola has an axis of symmetry passing through its vertex which is its turning point. If the quadratic has real roots, due to the symmetry of the parabola, the turning point lies halfway between the roots which are x-intercepts at.

In our case $x_1 = 7$ and $x_2 = -2$ so the midpoint is at $\frac{x_1 + x_2}{2}$

$$x_v = -\frac{b}{2a} = -\frac{-5}{2 \cdot 1} = \frac{5}{2}$$

The precise coordinates of the vertex are:

$$(x_v, y_v) = (-\tfrac{b}{2a}, \tfrac{D}{4a^2}) = (\tfrac{5}{2}, \tfrac{9}{4})$$

5. Solve the equation

$$x + \frac{1}{x} = 4.25$$

The first observation is that $x = 0$ cannot be a solution, so we can multiply with x. Then we have the equivalent equation

$$xx + \frac{1}{x}x = 4.25x$$
$$x^2 + 1 = 4.25x$$

Multiply both sides by 100 to get a simpler equivalent equation

$$100x^2 + 100 = 425x$$

or equivalent

$$100x^2 - 425x + 100 = 0$$

Applying the discriminant formula

$$x_1 = \frac{-(-425) \pm \sqrt{(-425)^2 - 4 \cdot\ 100 \cdot\ 100}}{2 \cdot\ 100 =} = \frac{-(-425) \pm\ 375}{2 \cdot\ 100} = 4, \frac{1}{4}$$

Review Vieta Formulas which allow us to find the sum and the product of roots of a quadratic equation without explicitly calculating them.

$$x_1 + x_2 = -\frac{b}{a}, \text{ and } x_1 x_2 = \frac{c}{a}$$

If $a = 1$, we have: $x_1 + x_2 = -b$ and $x_1 x_2 = c$

Why? If x_1, x_2 are the two roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

By rewriting we get

$$a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1 \cdot x_2$$

Two quadratics are equal if their coefficients are, so we get

$$b = -a(x_1 + x_2), \text{ and } c = ax_1 \cdot x_2$$

7. Let $S = x_1 + x_2 = 7$ and $P = x_1 \cdot x_2 = 8$

- Write down the quadratic equation so that x_1 and x_2 are its solutions.
- Calculate ${x_1}^2 + {x_2}^2$.
- Calculate $\frac{1}{x_1} + \frac{1}{x_2}$.

$$x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = x^2 - 7x + 8$$

To calculate ${x_1}^2 + {x_2}^2$ one has to express it in terms of S and P.
We know $(u+v)^2 = u^2 + 2uv + v^2$. Thus, $u^2 + v^2 = (u+v)^2 - 2uv = S^2 - 2P = 49 - 2 \cdot 8 = 49 - 16 = 33$
To calculate $\frac{1}{x_1} + \frac{1}{x_2}$ one has to express it in terms of S and P.

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1 \cdot x_2} = \frac{7}{8}$$