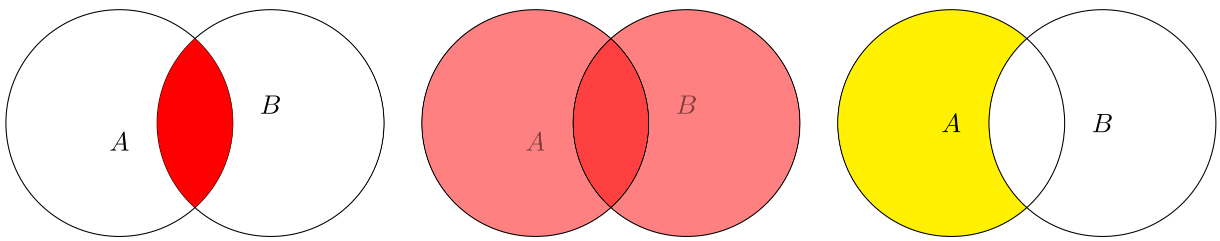
MATH 8: HANDOUT 7
LOGIC 1: FROM LOGIC TO PROOFS

Why logic?

A Venn diagram is a representation with overlapping circles. The interior of the circle symbolically represents the elements of the set, while the exterior represents elements that are not members of the set. Venn diagrams do not generally contain information on the relative or absolute sizes (cardinality) of sets.

Basic operations: Intersection of two sets $A \cap B$, Union of two sets $A \cup B$, Complement of "B" in "A" $B^c \cap A = A \setminus B$(i.e. "what is in A and not in B") Absolute complement of A in the universe U $A^c = U \setminus A$



Let us try to solve Lewis Carol's puzzle: All hummingbirds are richly colored. No large birds live on honey. Birds that do not live on honey are dull in color. Therefore ?. . . .

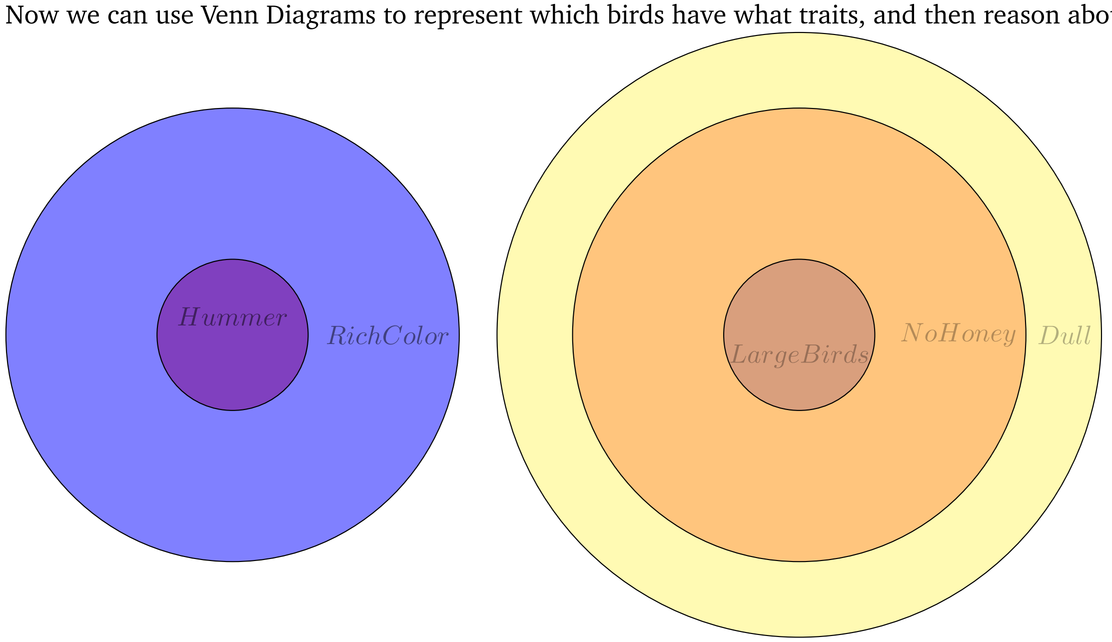
We should first reformulate these statements using an *if* **something**, followed by a *then* **something else**.

"All hummingbirds("hummers") are richly colored.": If $\underbrace{\text{a bird is a hummingbird}}_{A}$ then $\underbrace{\text{the bird is richly colored.}}_{B}$

"No large birds live on honey" : If $\underbrace{\text{a bird is large}}_{C}$ then $\underbrace{\text{the bird does not live on honey.}}_{D}$

"Birds that do not live on honey are dull in color." : If $\underbrace{\text{a bird does not live on honey}}_{D}$ then $\underbrace{\text{the bird is dull in color.}}_{E}$

Now we can use Venn Diagrams to represent which birds have what traits, and then reason about them.



A is entirely in B. E is altogether outside B. C is entirely in D. D is entirely in E.

Conclusion: C is therefore outside B. C is therefore outside A. A is therefore outside C. A is therefore outside D. A is therefore outside E.

Conclusion:

Large birds are not richly colored. Large birds are not hummingbirds. Hummers (hummingbirds) are small, richly colored, honey eating birds.

Basic logic operations

- NOT (for example, NOT A): true if A is false, and false if A is true. Commonly denoted by $\neg A$ or (in computer science) $!A$.
- AND (for example A AND B): true if both A, B are true, and false otherwise (i.e., if at least one of them is false). Commonly denoted by $A \wedge B$
- OR (for example A OR B): true if at least one of A, B is true, and false otherwise. Sometimes also called "inclusive or" to distinguish it from the "exclusive or" described in problem 4 below. Commonly denoted by $A \vee B$.

As in usual algebra, logic operations can be combined, e.g. $(A \vee B) \wedge C$.

Truth tables

If we have a logical formula involving variables $A, B, C, \dots$, we can make a table listing, for every possible combination of values of $A, B, \dots$, the value of our formula. For example, the following is the truth tables for OR and AND:

A	B	A OR B
T	T	T
T	F	T
F	T	T
F	F	F

A	B	A AND B
T	T	T
T	F	F
F	T	F
F	F	F

Logic laws

We can combine logic operations, creating more complicated expressions such as $A \wedge (B \vee C)$. As in arithmetic, these operations satisfy some laws: for example $A \vee B$ is the same as $B \vee A$. Here, "the same" means "for all values of A, B, these two expressions give the same answer"; it is usually denoted by $\iff$. Here are two other laws:

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$
$$A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$$

Truth tables provide the most straightforward (but not the shortest) way to prove complicated logical rules: if we want to prove that two formulas are equivalent (i.e., always give the same answer), make a truth table for each of them, and if the tables coincide, they are equivalent.

Problems

1. Write the truth table for each of the following formulas. Are they equivalent (i.e., do they always give the same value)?
   (a) $(A \vee B) \wedge (A \vee C)$
   (b) $A \vee (B \wedge C)$.

2. Use the truth tables to prove *De Morgan's laws*

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$
$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B)$$

3. Use truth tables to show that $\vee$ is commutative and associative:

$$A \vee B \iff B \vee A$$
$$A \vee (B \vee C) \iff (A \vee B) \vee C$$

Is it true that $\wedge$ is also commutative and associative?

4. Another logic operation, called "exclusive or", or XOR, is defined as follows: $A \text{ XOR } B$ is true if and only if exactly one of A, B is true.

(a) Write a truth table for XOR

(b) Describe XOR using only basic logic operations AND, OR, NOT, i.e. write a formula using variable A, B and these basic operations which is equivalent to $A \text{ XOR } B$.

5. Yet one more logic operation, NAND, is defined by

$$A \text{ NAND } B \iff \text{NOT}(A \text{ AND } B)$$

(a) Write a truth table for NAND

(b) What is $A \text{ NAND } A$?

*(c) Show that you can write NOT A, A AND B, A OR B using only NAND (possibly using each of A, B more than once).

This last part explains why NAND chips are popular in electronics: using them, you can build **any** logical gates.

6. Rewrite the following statements as if-then:

- All birds have wings.
- Two angles are supplementary if they are a linear pair.
- There are clouds in the sky if it is raining.

7. Find the logical conclusion of the statements below. People allergic to bananas are also allergic to oranges. Who is allergic to apples is not allergic to oranges.

8. On the island of knights and knaves, there are two kinds of people: Knights, who always tell the truth, and Knaves, who always lie. Unfortunately, there is no easy way of knowing whether a person you meet is a knight or a knave. . .

You meet two people on this island, Bart and Ted. Bart claims, "I and Ted are both knights or both knaves." Ted tells you, "Bart would tell you that I am a knave." So who is a knight and who is a knave?