LOGIC 2: LOGIC GATES AND LOGIC PROOFS
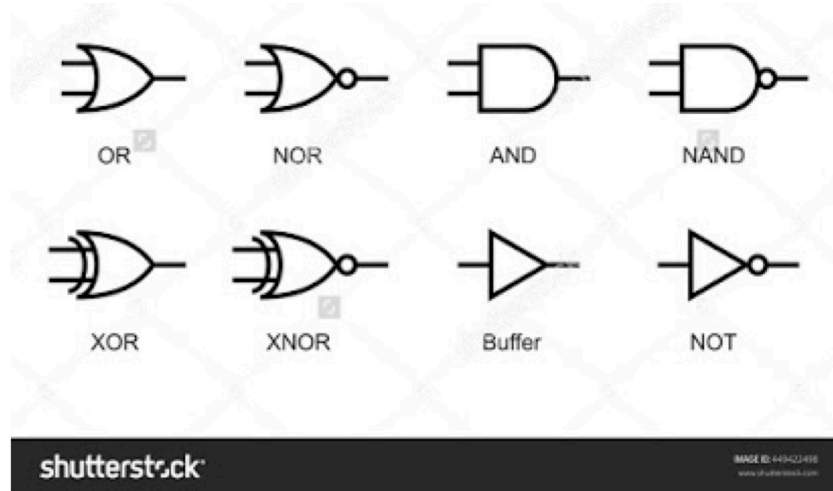
LOGIC GATES

LOGIC GATES IN ELECTRONICS

In electronics, it is common to have digital signals which take two values: LOW and HIGH (e.g., in certain microcontrollers, voltage can range between 0–5V, and anything over 2.5V is considered HIGH, and anything below, LOW). These two values can also be considered as two binary digits: LOW= 0, HIGH= 1, or as boolean values: LOW= 0 =false, HIGH= 1 =true.
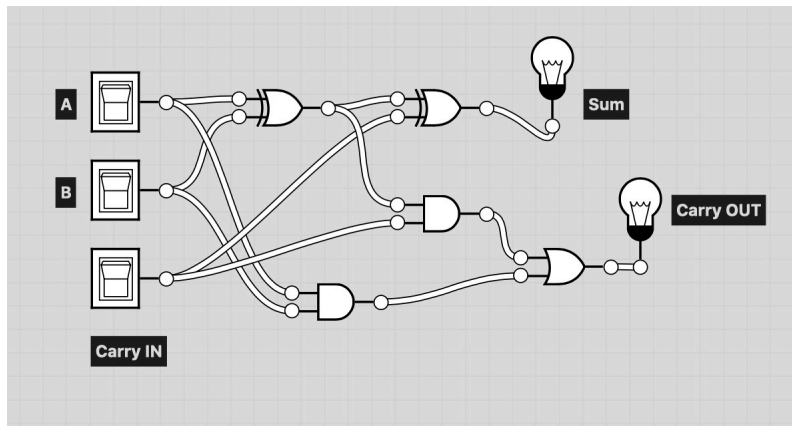
The basis of all modern computers are "logic gates", chips that take two (or more) such inputs and produce an output described by some truth table. (These chips contain transistors and diodes, but this is irrelevant for us). For example, below is a typical such chip, containing four AND gates, each with two inputs:

Combining such simple gates, one can create more complicated ones — and use that to create circuits which take as input a collection of binary digits and produce as output some function such as sum or product of inputs (interpreting n binary inputs as an n-digit binary number).

There is a number of online simulators that allow you to create and test such circuits virtually; in particular, we can use http://logic.ly/demo.

An example of circuit is a *full adder*, which has three inputs: two binary digits A and B and a carry, and has two outputs, the sum and the carry output:

Its structure can be seen as made of two half-adders, with the more significant digit joined with an OR into the carry output. A *half-adder* has two binary inputs A and B and produces two outputs representing the two binary digits of the sum.

PROOFS

Let us formalize the "IF" logic operation – "if A then B", written as $A \implies B$, and meaning if A is true, then this operation is true only when B is also true, and if A is false, the operation is always true.

The "IF AND ONLY IF" logic operation is written $A \iff B$ and it means that we have both $A \implies B$ and $B \implies A$.

What is a proof?

A common answer is: a sequence of statements, starting with given ones and ending in a statement which we want to prove, and such that each statement in the sequence *logically follows from the previous*. But what does "logically follows from the previous" mean ?

In the simple case when all our statements can be written as combinations of the same elementary statements (which we can denote by letters $A, B, \ldots$), using logical operations, it means the following:

> For any combinations of values of letters $A, B, \ldots$ which makes the previous statements true, the next statement is also true.

Thus, it can be checked simply by a truth table. E.g., statement $\neg A$ logically follows from $A \implies B$ and $\neg B$: in all cases when $A \implies B$ and $\neg B$ are true, $\neg A$ is also true, as is easy to check.

We can also check that $A \implies B$ is equivalent to $(\neg A) \vee B$.

However, usually instead of writing truth tables, people use some simple logical laws repeatedly. Here are some commonly used laws:

- Given $A \implies B$ and A, we can conclude B.
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! it only means that **if** A is true, then so is C.]
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$

PROBLEMS

1. Draw the circuit diagrams for NOT, AND and OR using NAND gates only.
2. Show that the NOT, AND and OR operations can be expressed using NOR, that is defined as: A NOR $B =$ NOT$(A$ OR $B)$. Draw the circuit diagrams for NOT, AND and OR using NOR gates only.
3. Show that for any value of A, expression $A \vee (\neg A)$ is true (such expressions, which are true for all values of variables involved, are called *tautologies*). This particular tautology is sometimes called **"law of excluded middle"** (meaning there is no middle ground — A must be true or false).
   Similarly, show that $A \wedge (\neg A)$ is always false.
4. Carefully draw the two sets, one included in the other, for the following two implications, and then think about one or two more common, day-to-day similar examples, write them up and do the same for them.
   (a) If a person is 18 years old then that person can vote.
   (b) If a bottle contains milk than that bottle is stored in the fridge.
5. A teacher tells the student "If you do not take the final exam, you get an F". Does it mean that
   (a) If the student does take the final exam, he will not get an F
   (b) If the student does not get an F, it means he must have taken the final exam.
6. If today is Thursday, then Jane's class has library day. If Jane's class has library day, then Jane will bring home new library books. Jane brought no new library books. Therefore,...
   Can you guess what would be the natural conclusion from these 3 statements? Can you prove it using some laws of logic? It might help to write each of them as combination of elementary statements e.g. T for "today is Tuesday", L for "Jane's class has a library day", etc.