MATH 8: HANDOUT 9
LOGIC 3: PROOFS CONTINUED, QUANTIFIERS

COMMONLY USED LAWS OF LOGIC

- Given $A \implies B$ and A, we can conclude B *(Modus Ponens)*
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! It only means that **if** A is true, then so is C.]
- Given $A \wedge B$, we can conclude A (and we can also conclude B)
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$ *(Modus Tollens)*
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ *(De Morgan Law)*
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ *(Law of contrapositive)*

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

COMMON METHODS OF PROOF

Proof by cases.

Example: Prove that for any integer n, the number $n(n+1)$ is even.

Proof. If n is integer, it is even or odd. If n is even, then $n(n+1)$ is even (a multiple of even is always even). If n is odd, then $n+1$ is even and thus $n(n+1)$ is even by same reasoning.

Thus, in all cases $n(n+1)$ is even. □

General scheme:

Given

$$A_1 \vee A_2$$
$$A_1 \implies B$$
$$A_2 \implies B$$

we can conclude that B is true.

You can have more than two cases.

Note: it is important to verify that the cases you consider cover all possibilities (i.e. that at least one of the statements A_1, A_2 is always true).

Conditional proof.

Example: Prove that if n is even, then n^2 is even.

Proof. Assume that n is even. Then $n^2 = n * n$ is also even, since a multiple of even is even. □

General scheme

To prove $A \implies B$, we can

- Assume A
- Give a proof of B (in the proof, we can use that A is true).

This proves $A \implies B$ (without any assumptions).

Proof by contradiction.

Example: Prove that if x is a real root of polynomial $p(x) = 10x^3 + 2x + 15$, then x must be negative.

Proof. Assume that x is not negative, i.e. $x \geq 0$. Then $p(x) = 10x^3 + 2x + 15 \geq 15$, which contradicts the fact that x is a root of $p(x)$. Thus, our assumption can not be true, so x must be negative. □

General scheme

To prove that A is true, assume A is false, and derive a contradiction. This proves that A must be true.

Quantifiers

Existential quantifier: To write statements of the form "There exists an x such that...", use existential quantifier:

$$\exists x \in A: \text{ (some statement depending on } x)$$

Here A is a set of all possible values of variable x.
Example: $\exists x \in \mathbb{R}: x^2 = 5$.

Note that following the quantifier, you must have a *statement*, i.e. something that can be true or false. Usually it is some equality or inequality. You can't write there an expression which gives numerical values (for example, $\exists x \in \mathbb{R}: x^2 + 1$) — it makes no sense.

Universal quantifier: To write statements of the form "For all values of x we have...", use universal quantifier:

$$\forall x \in A: \text{ (some statement depending on } x)$$

Here A is a set of all possible values of variable x.
Example: $\forall x \in \mathbb{R}: x^2 > 0$.

Logic proofs involving quantifiers

To prove a statement $\exists x \in A: \dots$, it suffices to give one example of x for which the statement denoted by dots is true. You need to verify that the statement is true for that value of x, but it is not necessary to explain how you found this value, nor is it necessary to find how many such values there are.
Example: to prove $\exists x \in \mathbb{R}: x^2 = 9$, take $x = 3$; then $x^2 = 9$.

To prove a statement $\forall x \in A: \dots$, you need to give an argument which shows that for any $x \in A$, the statement denoted by dots is true. **Considering one, two, or one thousand examples is not enough.**
Example: to prove $\forall x \in \mathbb{R}: x^2 + 2x + 4 > 0$, we could argue as follows. Let x be an arbitrary real number. Then $x^2 + 2x + 4 = (x+1)^2 + 3$. Since a square of a real number is always non-negative, $(x+1)^2 \geq 0$, so $x^2 + 2x + 4 = (x+1)^2 + 3 \geq 0 + 3 > 0$.

Note that this argument works for any x; it uses no special properties of x except that x is a real number.

De Morgan laws for quantifiers

(Assuming that A is a nonempty set).

$$\neg\Big(\forall x \in A: P(x)\Big) \iff \Big(\exists x \in A: \neg P(x)\Big)$$
$$\neg\Big(\exists x \in A: P(x)\Big) \iff \Big(\forall x \in A: \neg P(x)\Big)$$

For example, negation of the statement "All flowers are white" is "There exists a flower which is not white", or in more human language, "Some flowers are not white".

Combining techniques:

Prove that $\sqrt{2}$ is not a rational number. To help, we first prove that for any natural number a, if a^2 is even, then a itself is also even. Then we use that fact, together with the fact that any rational number can be written as a pair of two natural numbers which have no common divisor, to arrive at a contradiction: we assume that $\sqrt{2}$ is rational, and thus there exist two natural numbers a and b which have no common divisor, and conclude that they must have 2 as a common divisor.

Problems

1. The following statement is sometimes written on highway trucks:
If you can't see my mirrors, I can't see you.
Can you write an equivalent statement without using word "not" (or its variations such as "can't").

2. Write the following statements using quantifiers:
(a) All birds can fly
(b) Not all birds can fly
(c) Some birds can fly
(d) All large birds can fly

(e) Only large birds can fly
(f) No large bird can fly

You can use letter B for the set of all birds, and notation $F(x)$ for statement "x can fly" and $L(x)$ for "x is large".

3. Write the following statements using logic operations and quantifiers, and the notation

- P for the set of all people,
- $M(x)$ to say that x is a mathematician,
- $L(x)$ to say that x loves music
- $J(x)$ to say that x goes to John's party

(a) All mathematicians love music
(b) Some mathematicians don't like music
(c) No one but a mathematician likes music
(d) No one would go to John's party unless he loves music or is a mathematician

4. Write each of the following statements using only quantifiers, arithmetic operations, equalities and inequalities. In all problems, letters x, y, z stand for a variables that takes real values, and letters $m, n, k, \dots$ stand for variables that take integer values.

(a) Equation $x^2 + x - 1$ has a solution
(b) Inequality $y^3 + 3y + 1 < 0$ has a solution
(c) Inequality $y^3 + 3y + 1 < 0$ has a positive real solution
(d) Number 100 is even.
(e) Number 100 is odd
(f) For any integer number, if it is even, then its square is also even.

5. For each of the statements of the previous problem, try to determine if it is true. If it is, give a proof. If not, disprove it (i.e., prove its negation).

6. Prove that for any integer number n, the number $n(n+1)(2n+1)$ is divisible by 3. Is it true that such a number must also be divisible by 6?

You can use without proof the fact that any integer can be written in one of the forms $n = 3k$ or $n = 3k + 1$ or $n = 3k + 2$, for some integer k.

7. A function $f(x)$ is called *monotonic* if $(x_1 < x_2) \implies (f(x_1) < f(x_2))$. Prove that a monotonic function can't have more than one root. *[Hint: use assume that it has at least two distinct roots and derive a contradiction.]*

8. Prove that $\sqrt{3}$ is not a rational number, along the same lines as it was done for $\sqrt{2}$, by first proving a similar helping statement: if the square of a number is a multiple of 3, then the number itself must be a multiple of 3. Could we do the same for 5, or for 7, and so on ? What would be the "so on" here ?

9. Prove by contradiction that there does not exist a smallest positive real number.