MATH 8: HANDOUT 12

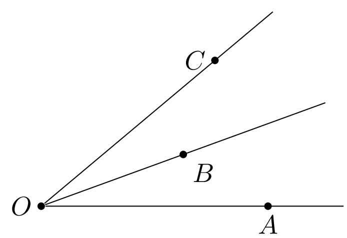
EUCLIDEAN GEOMETRY 2: FIRST THEOREMS. PARALLEL LINES. TRIANGLES.

1. Axioms – just to recap

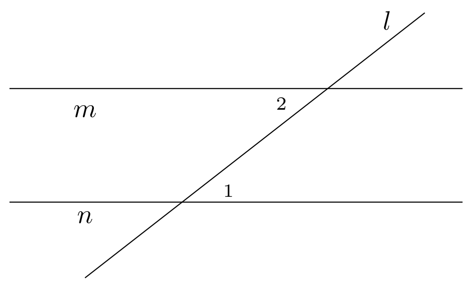
Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



2. First theorems

Now we can proceed with proving some results based on the axioms above.

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Proof. Proof by contradiction: Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more than one point) must be false. □

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Proof. Here we have to prove two things: the existence of a parallel line through the given point not on the given line, and its uniqueness. Below we provide a sketch of the proof – please fill in the details and draw a diagram at home!

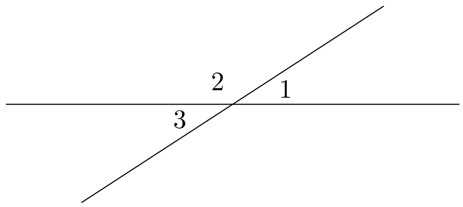
Existence: Let m be any line that goes through P and intersect l at point O. Let A be a point on the line l. Then we can measure the angle $\angle POA$. Now, let PB be such that $m\angle BPO = m\angle POA$ and B is on the other side of m than A. In this case, by Axiom 4, $\overleftrightarrow{PB} \parallel l$.

Uniqueness: Imagine that there are two lines m, n that are parallel to l and go through P. Take a line k that goes through P and intersects l in point O. Let A be a point on line l distinct from O, and B, C — points on lines m and n respectively on the other side of line k than A. Since both m, n are parallel to l, we can see that $m\angle AOP = m\angle BPO = m\angle CPO$ – but that would mean that lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{CP}$ are the same — contradiction to our assumption that there are two such lines. □

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Proof. Assume that l and n are not parallel and intersect at point P. But then it appears that there are two lines that are parallel to m are go through point P — contradiction with Theorem 2. □

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.*



Proof. Let angle 2 be as shown in the figure to the left. Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are* perpendicular *and write $l \perp m$.)*

Proof. Left as a homework exercise. □

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Proof. Left as a homework exercise. □

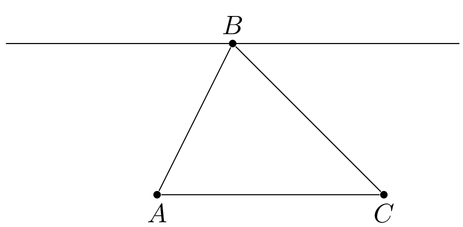
Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Proof. Left as a homework exercise. □

3. Triangles

Theorem 8. *Given any three points A, B, C, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a* triangle*, written $\triangle ABC$. The three respective angles are called the triangle's* interior angles*.)*

Proof. The proof is based on the figure below and use of Alternate Interior Angles axiom. Details are left to you as a homework.



□

4. Congruence

It will be helpful, in general, to have a way of comparing geometric objects to tell whether they are the same. We will build up such a notion and call it congruence of objects. To begin, we define congruence of angles and congruence of line segments (note that an angle cannot be congruent to a line segment; the objects have to be the same type).

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \implies \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

5. Congruence of Triangles

Triangles consist of six pieces (three line segments and three angles), but some notion of constancy of shape in triangles is important in our geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we can take them as axioms and continue.

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

6. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

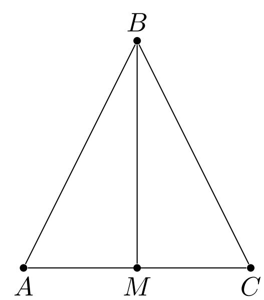
Theorem 9 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

A proof is given to you as homework.

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC (it exists and is unique by Theorem 7 about the existence of the perpendicular); the median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

Theorem 10. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*



Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and side BM is the same in both triangles. Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.
Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. □

Homework

1. **1.** (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 4 will be of key importance.
   - (a) Study the proof of Theorem 2 and draw a diagram that illustrates it.
   - (b) Study the proof of Theorem 3.
   - (c) Prove Theorem 5.
   - (d) Prove Theorem 6.
   - (e) Prove Theorem 7.

2. Complete the proof of Theorem 8, about sum of angles of a triangle.

3. What is the sum of angles of a quadrilateral? of a pentagon?

4. Notice that SSA and AAA are not listed as congruence rules.

(a) Describe a pair of triangles that have two congruent sides and one congruent angle but are not congruent triangles.

(b) Describe a pair of triangles that have three congruent angles but are not congruent triangles.

5. Prove that the following two properties of a triangle are equivalent:

(a) All sides have the same length.

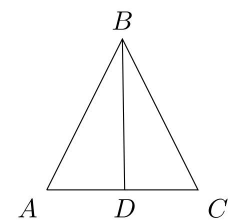
(b) All angles are $60°$.

A triangle satisfying these properties is called *equilateral*.

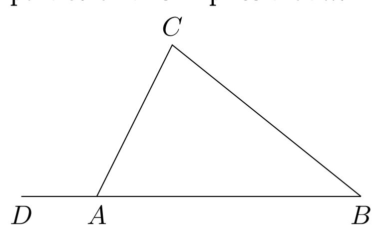
6. A triangle in which two sides are congruent is called *isosceles*. Such triangles have many special properties.

(a) Let $\triangle ABC$ be an isosceles triangle, with $\overline{AB} \cong \overline{BC}$. Suppose D is a point on $\overline{AC}$ such that $\overline{AD} \cong \overline{DC}$ (such point is called *midpoint* of the segment). Prove that then, $\triangle BD \cong \triangle CBD$ and deduce from this that $\angle DBA \cong \angle DBC$, and $\angle A \cong \angle C$. What can we say about $\angle ADB$?

(b) Conversely, show that if $\triangle ABC$ is such that $\angle A \cong \angle C$, then $\triangle ABC$ is isosceles, with $\overline{AB} \cong \overline{BC}$.



7. Given a triangle $\triangle ABC$, let D be a point on the line AB, so that A is between D and B. In this situation, angle $\angle DAC$ is called an *external angle* of $\triangle ABC$. Prove that $m\angle DAC = m\angle B + m\angle C$ (in particular this implies that $m\angle DAC > m\angle B$, and similarly for $\angle C$).



8. (Perpendicular bisector) Let $\overline{AB}$ be a line segment. The perpendicular bisector L of $\overline{AB}$ is the line that passes through the midpoint M of $\overline{AB}$ and is perpendicular to $\overline{AB}$.

(a) Prove that for any point P on L, triangles $\triangle APM$ and $\triangle BPM$ are congruent. Deduce from this that $AP = BP$.

(b) Conversely, let P be a point on the plane such that $AP = BP$. Prove that then P must be on L.

Taken together, these two statements say that a point P is *equidistant* from A, B if and only if it lies on the perpendicular bisector L of segment $\overline{AB}$. Another way to say it is that the *locus* of all the points equidistant from A, B is the perpendicular bisector of $\overline{AB}$.

9. Show that for any triangle $\triangle ABC$, the perpendicular bisectors of the three sides intersect at a single point, and this point is equidistant from all three vertices of the triangle. [**Hint**: consider the point where two of the bisectors intersect. Prove that this point is equidistant from all three vertices.]

Note: the intersection point can be outside the triangle.