MATH 8: HANDOUT 13
EUCLIDEAN GEOMETRY 3: TRIANGLE INEQUALITIES.

8. Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 9). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

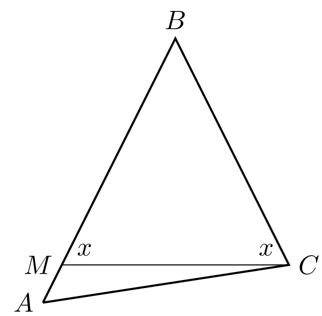
Theorem 11. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 9, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$ (these two angles are denoted by x in the figure.) On one hand, $m\angle C > x$ (this easily follows from Axiom 3). On the other hand, since x is an external angle of $\triangle AMC$, by Problem 6 from Handout 12, we have $x > m\angle A$. These two inequalities imply $m\angle C > m\angle A$, which contradicts what we started with.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction.



□

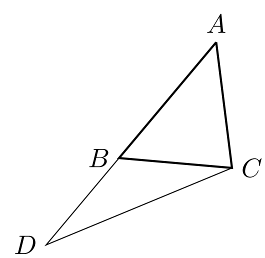
The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

Theorem 12. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 13 (The triangle inequality). *In $\triangle ABC$, we have $AB + BC > AC$.*

Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 11, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □



Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. (Slant lines and perpendiculars) Let P be a point not on line l, and let $Q \in l$ be such that $PQ \perp l$. Prove that then, for any other point R on line l, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.

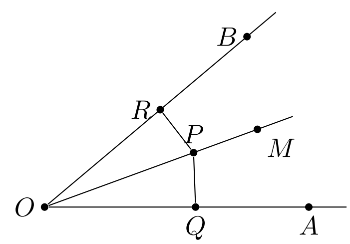
Note: you can not use the Pythagorean theorem for this, as we haven't yet proved it! Instead, use Theorem 11.

2. (Angle bisector). Define a distance from a point P to line l as the length of the perpendicular from P to l (compare with the previous problem).

Let $\overrightarrow{OM}$ be the angle bisector of $\angle AOB$, i.e. $\angle AOM \cong \angle MOB$.

(a) Let P be any point on $\overrightarrow{OM}$, and PQ, PR – perpendiculars from P to sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. Use ASA axiom to prove that triangles $\triangle OPR$, $\triangle OPQ$ are congruent, and deduce from this that distances from P to $\overrightarrow{OA}$, $\overrightarrow{OB}$ are equal.

(b) Prove that conversely, if P is a point inside angle $\angle AOB$, and distances from P to the two sides of the angle are equal, then P must lie on the angle bisector $\overrightarrow{OM}$



These two statements show that the locus of points equidistant from the two sides of an angle is the angle bisector

3. Prove that in any triangle, the three angle bisectors intersect at a single point (compare with the similar fact about perpendicular bisectors)