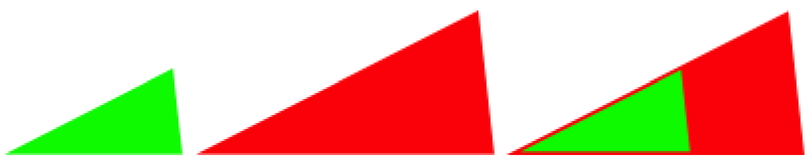
October 30, 2022

Geometry.

Similarity and homothety.

<u>**Recap: Similar triangles**</u>

Definition. Two triangles are similar if (i) angles of one of them are congruent to the respective angles of the other, or (ii) the sides of one of them are proportional to the homologous sides of the other.



Arranging 2 similar triangles, so that the intercept theorem can be applied

The similarity is closely related to the intercept (Thales) theorem. In fact this theorem is equivalent to the concept of similar triangles, i.e. it can be used to prove the properties of similar triangles, and similar triangles can be used to prove the intercept theorem. By matching identical angles one can always place 2 similar triangles in one another, obtaining the configuration in which the intercept theorem applies and vice versa the intercept theorem configuration always contains 2 similar triangles. In particular, a line parallel to any side of a given triangle cuts off a triangle similar to the given one.

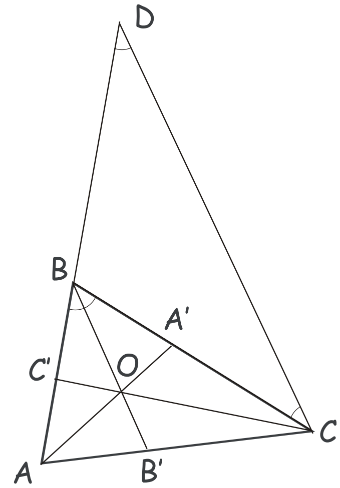
Similarity tests for triangles.

- Two angles of one triangle are respectively congruent to the two angles of the other
- Two sides of one triangle are proportional to the respective two sides of the other, and the angles between these sides are congruent
- Three sides of one triangle are proportional to three sides of the other

Application: property of the bisector.

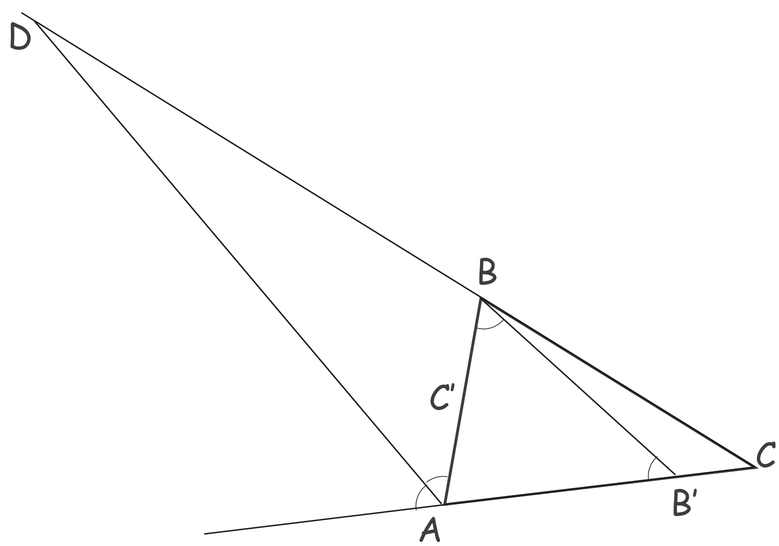
Theorem (property of the bisector). The bisector of any angle of a triangle divides the opposite side into parts proportional to the adjacent sides,

$\frac{|AC'|}{|C'B|} = \frac{|AC|}{|BC|}, \frac{|BA'|}{|A'C|} = \frac{|AB|}{|AC|}, \frac{|CB'|}{|B'A|} = \frac{|BC|}{|AB|}$



Proof. Consider the bisector BB'. Draw line parallel to BB' from the vertex C, which intercepts the extension of the side AB at a point D. Angles $B'BC$ and BCD have parallel sides and therefore are congruent. Similarly, are congruent ABB' and CDB. Hence, triangle CBD is isosceles, and $|BD| = |BC|$. Now, applying the intercept theorem to the triangles ABB' and ACD, we obtain $\frac{|CB'|}{|B'A|} = \frac{|BD|}{|AB|} = \frac{|BC|}{|AB|}$.

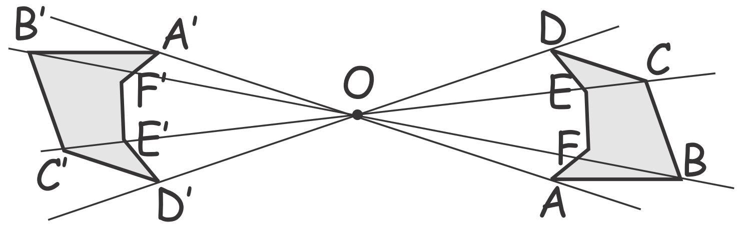
Theorem (property of the external bisector). The bisector of the exterior angle of a triangle intercepts the opposite side at a point (D in the Figure) such that the distances from this point to the vertices of the triangle belonging to the same line are proportional to the lateral sides of the triangle.



Proof. Draw line parallel to AD from the vertex B, which intercepts the side AC at a point B'. Angles ABB' and DAB have parallel sides and therefore are congruent. Similarly, we see that angles AC'B and ABB' are congruent, and, therefore, |AB'| = |AB|. Applying the intercept theorem, we obtain, $\frac{|DB|}{|DC|} = \frac{|AB'|}{|AC|} = \frac{|AB|}{|AC|}$.

Recap: Central Symmetry.

Central symmetry



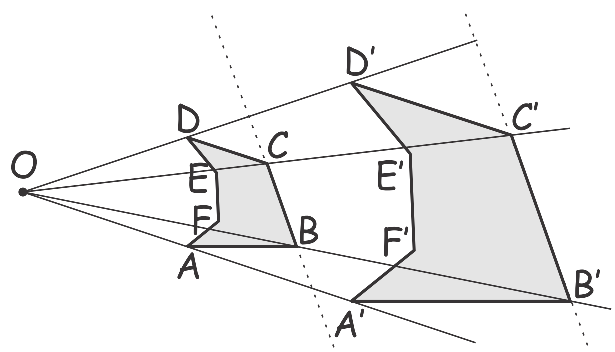
Definition. Two points A and A' are symmetric with respect to a point O, if O is the midpoint of the segment AA'.

Definition. Two figures are symmetric with respect to a point O, if for each point of one figure there is a symmetric point belonging to the other figure, and vice versa. The point O is called the center of symmetry.

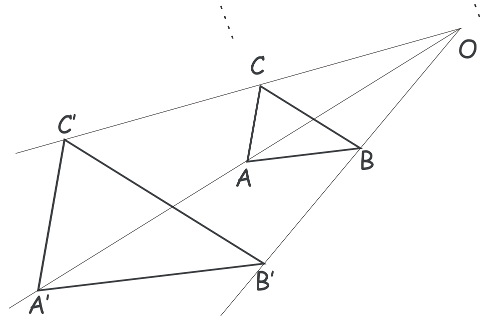
Symmetric figures are congruent and can be made to coincide by a 180 degree rotation of one of the figures around the center of symmetry.

Homothety.

Homothety



Definition. Two figures are homothetic with respect to a point O, if for each point A of one figure there is a corresponding point A' belonging to the other figure, such that A' lies on the line (OA) at a distance $|OA'| = k|OA|$ $(k > 0)$ from point O, and vice versa, for each point A' of the second figure there is a corresponding point A belonging to the first figure, such that A' lies on the line (OA) at a distance $|OA| = \frac{1}{k}|OA'|$ from point O. Here the positive number k is called the homothety (or similarity) coefficient. Homothetic figures are **similar**. The transformation of one figure (e.g. multilateral $ABCDEF$) into the figure $A'B'C'D'E'F'$ is called homothety, or similarity transformation.



Thales Theorem Corollary 1. The corresponding segments (e.g. sides) of the homothetic figures are parallel.

Thales Theorem Corollary 2. The ratio of the corresponding elements (e.g. sides) of the homothetic figures equals k.

Thales Theorem Corollary 3. If two triangles are homothetic to each other, then they are similar.

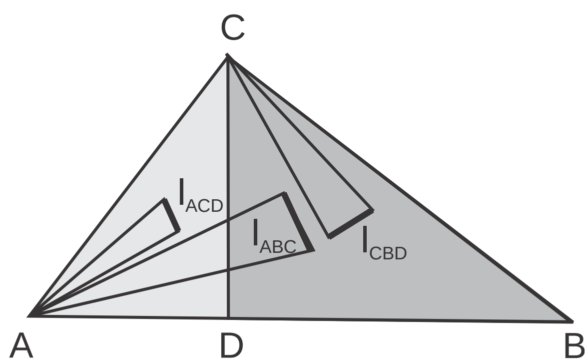
This can be used to define the notion of similarity for figures other than triangles.

Definition. Consider triangles, or polygons, such that angles of one of them are congruent to the respective angles of the other(s). Sides which are adjacent to the congruent angles are called *homologous*. In triangles, sides opposite to the congruent angles are also homologous.

Exercise. What is the ratio of the areas of two similar (homothetic) figures?

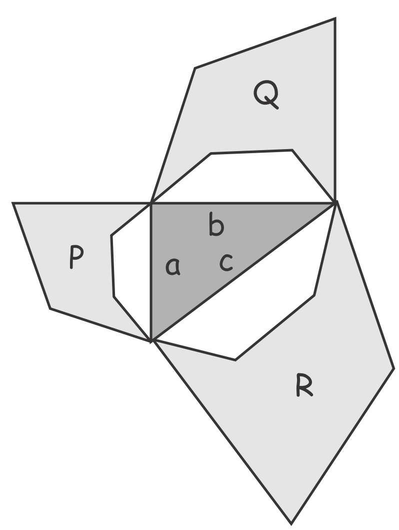
Generalized Pythagorean Theorem.

Theorem 1. For three homologous segments, l_{ABC}, l_{CBD} and l_{ACD} belonging to the similar right triangles ABC, CBD and ACD, where CD is the altitude of the triangle ABC drawn to its hypotenuse AB, the following holds,



$$l_{ACD}^2 + l_{CBD}^2 = l_{ABC}^2$$

Proof. If we square the similarity relation for the homologous segments, $\frac{l_{CBD}}{a} = \frac{l_{ACD}}{b} = \frac{l_{ABC}}{c}$, where $a = |BC|$, $b = |AC|$ and $c = |AB|$ are the legs and the hypotenuse of the triangle ABC, we obtain, $\frac{l_{CBD}^2}{a^2} = \frac{l_{ACD}^2}{b^2} = \frac{l_{ABC}^2}{c^2}$. Using the property of a proportion, we may then write, $\frac{l_{ACD}^2 + l_{CBD}^2}{a^2+b^2} = \frac{l_{ABC}^2}{c^2}$, wherefrom, by Pythagorean theorem for the right



triangle ABC, $a^2+b^2=c^2$, we immediately obtain $l_{ACD}^2+l_{CBD}^2=l_{ABC}^2$.

Theorem 2. If three similar polygons, P, Q and R with areas S_P, S_Q and S_R are constructed on legs a, b and hypotenuse c, respectively, of a right triangle, then,

$$S_P+S_Q=S_R$$

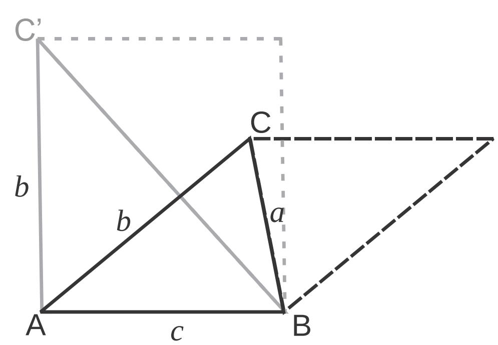
Proof. The areas of similar polygons on the sides of a right triangle satisfy $\frac{S_R}{S_P}=\frac{c^2}{a^2}$ and $\frac{S_R}{S_Q}=\frac{c^2}{b^2}$, or, $\frac{S_P}{a^2}=\frac{S_Q}{b^2}=\frac{S_R}{c^2}$. Using the property of a proportion, we may then write, $\frac{S_P+S_Q}{a^2+b^2}=\frac{S_R}{c^2}$, wherefrom, using the Pythagorean theorem for the right triangle, $a^2+b^2=c^2$, we immediately obtain $S_P+S_Q=S_R$.

Exercise. Show that for any proportion,

$$\left(\frac{a}{b}=\frac{c}{d}\right)\Rightarrow\left(\frac{a+c}{b+d}=\frac{a}{b}=\frac{c}{d}\right)\wedge\left(\frac{a-c}{b-d}=\frac{a}{b}=\frac{c}{d}, if\ b\neq d\right)$$

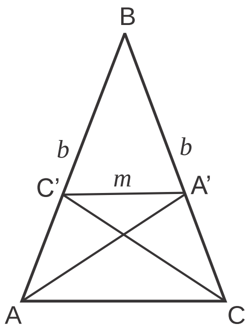
<u>**Selected problems on similar triangles (from last homeworks)**</u>.

Problem 1. Prove that for any triangle ABC with sides a, b and c, the area, $S\le\frac{1}{4}(b^2+c^2)$.



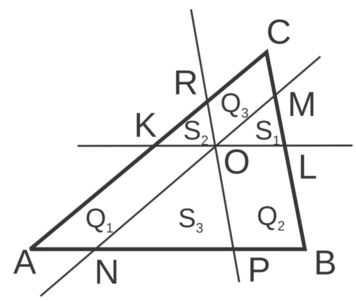
Solution. Notice that of all triangles with given two sides, b and c, the largest area has triangle ABC', where the sides with the given lengths, $|AB|=c$ and $|AC|=b$ form a right angle, $\widehat{BAC}=90°$ (b is the largest possible altitude to side c). Therefore, $\forall\Delta ABC, S_{ABC}\le S_{ABC'}=\frac{1}{2}bc\le\frac{1}{2}\frac{b^2+c^2}{2}$, where the last inequality follows from the arithmetic-geometric mean inequality, $bc\le\frac{b^2+c^2}{2}$ (or, alternatively, follows from $b^2+c^2-2bc=(b-c)^2\ge 0$.

Problem 2. In an isosceles triangle ABC with the side $|AB| = |BC| = b$, the segment $|A'C'| = m$ connects the intersection points of the bisectors, AA' and CC' of the angles at the base, AC, with the corresponding opposite sides, $A' \in BC$ and $C' \in AB$. Find the length of the base, $|AC|$ (express through given lengths, b and m).



Solution. From Thales proportionality theorem we have, $\frac{|AC|}{m} = \frac{|BC|}{|BA'|} = \frac{|BA'|+|A'C|}{|BA'|} = 1 + \frac{|A'C|}{|BA'|} = 1 + \frac{|AC|}{b}$, where we have used the property of the bisector, $\frac{|A'C|}{|BA'|} = \frac{|AC|}{|AB|} = \frac{|AC|}{b}$. We thus obtain, $|AC| = \frac{1}{\frac{1}{m}-\frac{1}{b}} = \frac{bm}{b-m}$.

Problem 5. Three lines parallel to the respective sides of the triangle ABC intersect at a single point, which lies inside this triangle. These lines split the triangle ABC into 6 parts, three of which are triangles with areas S_1, S_2, and S_3. Show that the area of the triangle ABC, $S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$ (see Figure).



Solution. Denote $\frac{S_1}{S} = k_1, \frac{S_2}{S} = k_2, \frac{S_3}{S} = k_3$. Then, $\frac{S_1+S_2+Q_3}{S} = k_1 + k_2 + \frac{Q_3}{S} = \left(\sqrt{k_1} + \sqrt{k_2}\right)^2$, so, $Q_3 = 2S\sqrt{k_1 k_2} = \sqrt{S_1 S_2}$, $Q_2 = \sqrt{S_3 S_1}$, $Q_1 = \sqrt{S_2 S_3}$.