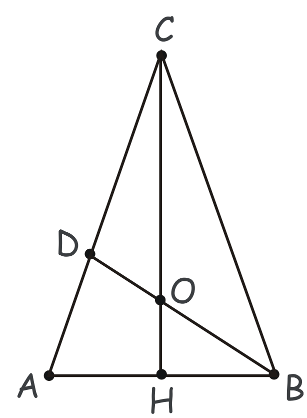
Homework for November 20, 2022

Geometry.

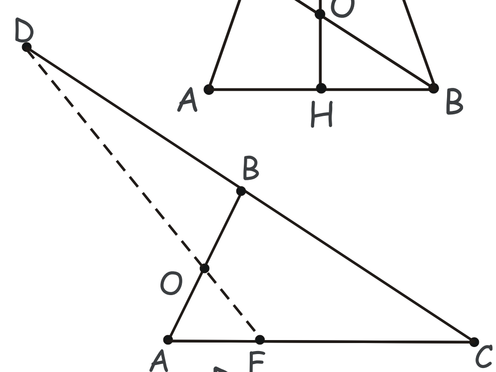
Review the classwork handout. Solve the unsolved problems from previous homeworks. Try solving the following problems, some of which we solved in the past using the similarity of triangles and Thales theorem, now using the method of point masses and the Law of Lever.

Problems.

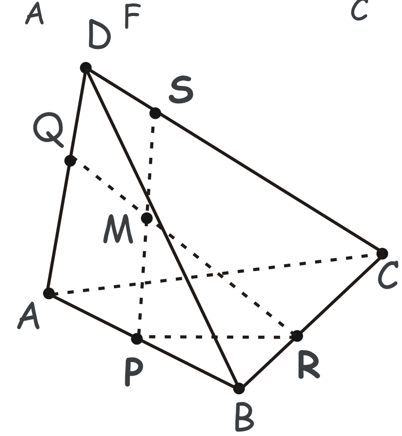
1. Prove that if a polygon has several axes of symmetry, they are all concurrent (cross at the same point).
2. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).
3. In isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.



4. Point D belongs to the continuation of side CB of the triangle ABC such that $|BD| = |BC|$. Point F belongs to side AC, and $|FC| = 3|AF|$. Segment DF intercepts side AB at point O. Find the ratio $|AO|:|OB|$.

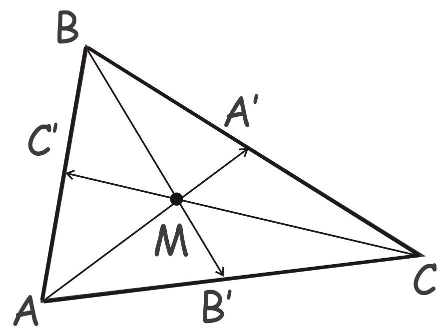


5. Consider segments connecting each vertex of the tetrahedron $ABCD$ with the centroid of the opposite face (the crossing point of its medians). Prove that all four of these segments, as well as the segments connecting the midpoints of the opposite edges (opposite edges have no common points; there are three pairs of opposite edges in a tetrahedron, and therefore three such segments) – seven segments in total, have common crossing point (are concurrent).



6. In a quadrilateral $ABCD$, E and F are the mid-points of its diagonals, while O is the point where the midlines (segments conneting the midpoints of the opposite sides) cross. Prove that E, F, and O are collinear (belong to the same line).

7. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).



8. What is the ratio of the two segments into which a line passing through the vertex A and the middle of the median BB' of the triangle ABC divides the median CC'?

9. In a parallelogram $ABCD$, a line passing through vertex D passes through a point E on the side AB, such that $|AE|$ is $1/n$-th of $|AB|$, n is an integer. At what distance from A, relative to the length, $|AC|$, of the diagonal AC it meets this diagonal?

Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework (you may skip the ones considered in class). Solve the following problems.

1. Using the results of the previous homework where you used the Euclidean algorithm to provide the continued fraction representation for the following numbers, construct and solve the Diophantine equations relating the numerator, denominator, and their GCD in the right-hand side.
   a. $\frac{1351}{780}$
   b. $\frac{25344}{8069}$
   c. $\frac{29376}{9347}$
   d. $\frac{6732}{1785}$
   e. $\frac{2187}{2048}$
   f. $\frac{3125}{2401}$

2. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

3. Find the value of the continued fraction given by
   a. $x = \{1,1,1,1,\dots\} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$
   b. $x = \{1,2,2,2,\dots\} = \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$
   c. $\{1,2,3,3,3,\dots\} = 1 + \frac{1}{2+\frac{1}{3+\frac{1}{3+\frac{1}{3+\cdots}}}}$

4. Find the set of all values of x for which the following expression makes sense: $\sqrt{25 - x^2} + \frac{4}{x-2} - \frac{1}{x}$.