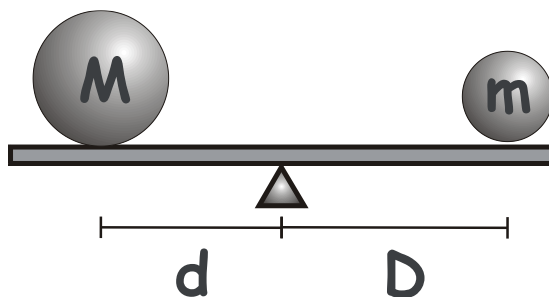November 20, 2022

# Geometry.

**The Method of the Center of Mass (mass points): Solving problems using the Law of Lever (mass points). Extended Ceva theorem. Menelaus theorem.**



**Theorem (Law of Lever).** Masses (weights) balance at distances from the fulcrum, which are inversely proportional to their magnitudes,

$$\frac{D}{d} = \frac{M}{m} \Leftrightarrow Md = mD$$

For commensurate masses, $= p \cdot w, m = q \cdot w, \; p, q \in \mathbb{N}$, the Law was proven using the main "trick" of the mass points method: each of the two masses is split into $2p$ and $2q$ smaller masses, $w/2$, respectively, which are then re-positioned in pairs around the original masses so that positions of the center of mass (COM) for each of the two original masses do not change, but the COM position for the whole system becomes obvious.

In order to prove the Law of Lever for incommensurate masses, we first make the following observation.

**Lemma**. If two commensurate masses $m$ and $M$ are placed at distances $D$ and $d$ from the fulcrum, respectively, then $M$ goes up if and only if $Md < mD$,

$$(M \; rises \; up) \Leftrightarrow (Md < mD)$$

First, if distances $d$ and $D$ are incommensurate, we move mass $M$ slightly, to a position $d'$ which is commensurate with $D$, but such that $M$ still rises up. Therefore, we only need to consider case when $d$ and $D$ are commensurate.

Since $M$ rises up, we need to increase mass $M$ to achieve balance. Let $M' > M$ be such that $M'$ and $m$ balance. Using the Law of Lever for commensurate masses we have, $M' = m\frac{D}{d}$ (because distances are commensurate, so are the masses). Since $M < M' = m\frac{D}{d}$, it follows that $Md < mD$. Conversely, if $Md < mD$ we can increase it to $M' = m\frac{D}{d}$, which balances $m$. Decreasing mass from back to $M$ will cause it to rise.

**Corollary**. The converse statement immediately follows via excluded middle,

$$(M \; goes \; down) \Leftrightarrow (Md > mD)$$

**Proof (case of incommensurate masses)**. Let now two incommensurate masses $m$ and $M$, be placed at distances $d$ and $D$ from the fulcrum, respectively, such that the Law of Lever is satisfied, $Md = mD$. Assume that the masses nevertheless do not balance, for example, $M$ goes down. Decrease mass $M$ by a small amount, turning it into $M'$, such that it still goes down, but is now commensurate with $m$. Now $m$ and $M'$ are commensurate, and $mD > M'd$, which means that $M'$ should rise. This contradicts our assumption, so $m$ and $M$ must balance. Note that in the above we used a non-trivial fact that a commensurate mass, or distance can be found that differs from the given incommensurate one by an arbitrarily small amount. This means that for any irrational number there exists a rational number, which differs from it as little as we want, i. e. that rational numbers are dense.

Solving problems using the Law of Lever.

For the objects in the uniform gravitational field, the Center of Gravity and the Center of Mass are equivalent. Archimedes uses the concept by considering bodies with the uniform density and defining the Center of Gravity based on postulated properties.

## Heuristic Definitions of the Center of Mass (Center of Gravity) known to Greeks.

1. The point such that if suspended at it, an object will remain motionless in the equilibrium, independent of the position that it is placed.
2. The point common to all the lines passing through the point at which the object is suspended
3. The point common to all lines on which the object balances.
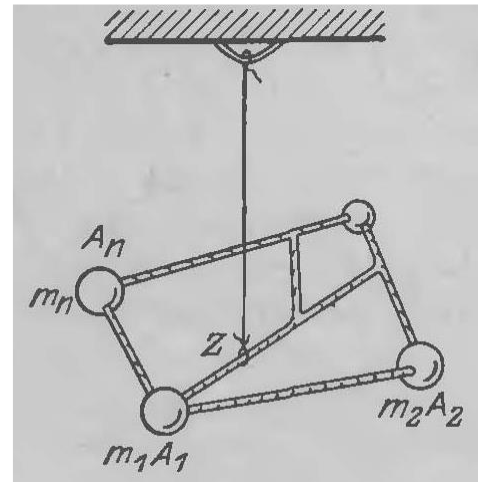


## Archimedes' postulates on the properties of the Center of Gravity (COM).

1. The COM of similar figures are similarly situated.
2. The COM of a convex figure lies within the figure.
3. If an object is cut in two pieces, then its COM lies on the line joining the COM's of the pieces, and its position satisfies the Law of Lever.



However, the situation is much simpler if we only consider point masses.

## Properties of the Center of Mass for a system of point masses.

1. Every system of finite number of point masses has unique center of mass (COM).

2. For two point masses, $m_1$ and $m_2$, the COM belongs to the segment connecting these points; its position is determined by the Archimedes lever rule: the point's mass times the distance from it to the COM is the same for both points, $m_1 d_1 = m_2 d_2$.
3. The position of the system's center of mass does not change if we move any subset of point masses in the system to the center of mass of this subset. In other words, we can replace any number of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.

## Solving problems using the COM.

Given a system of points and lines, one can derive various relations, such as concurrence of particular lines connecting some of the points, or the ratio of the lengths of different segments by associating certain masses with these points (i.e. placing point masses at their positions) and considering the center of mass of the obtained system of mass points.

**Exercise.** Prove that the medians of an arbitrary triangle $ABC$ are concurrent (cross at the same point $M$).
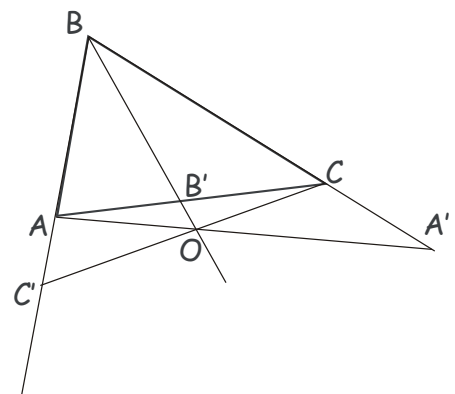
**Exercise.** Prove that the bisectors of an arbitrary triangle $ABC$ are concurrent (cross at the same point $O$).
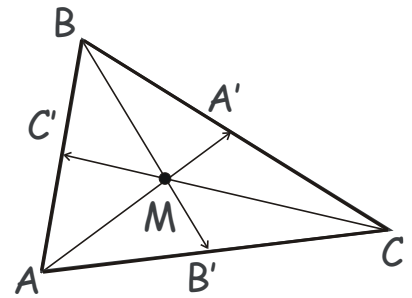
## Extended Ceva and Menelaus theorem.

**Theorem (Extended Ceva).** Segments (Cevians) connecting vertices $A$, $B$ and $C$, with points $A'$, $B'$ and $C'$ on the sides, or on the lines that suitably extend the sides $BC$, $AC$, and $AB$, of triangle $ABC$, are concurrent if and only if,

$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = 1$$

**Proof.** We have already proven this theorem for the case when points $A'$, $B'$ and $C'$ lie on the sides, but not on the lines extending the sides as it is shown in

the figure. Let us now consider this latter case. Let us first load points $A'$, B and $C'$ with masses $m_{A'}$, $m_B$ and $m_{C'}$, such that point $A$ is the center of mass for $m_B$ and $m_{C'}$, $m_B|AC'| = m_{C'}|AB|$, and point $C$ is the COM for $m_{A'}$ and $m_B$, $m_{A'}|BC| = m_B|A'C|$. Then, the COM of all three masses at the vertices of the triangle $A'BC'$ is at the point $O$, which is the intersection of $AA'$ and $CC'$. Let $BO$ cross side $AC$ at point $B'$. Adding mass to vertex $B$ would move the COM of the three masses along line $BO$, because the COM of the initial 3 masses is at $O$. Let us add another mass $m_B$ to vertex B, so that the total mass at this vertex is $2m_B$.The resulting system of masses then has the same COM as two masses, $m_B + m_{A'}$ and $m_B + m_{C'}$, at points $A$ and $C$, respectively. This COM is common to $AC$ and $BO$, and therefore is at point $B'$, so $(m_B + m_{A'})|AB'| = (m_B + m_{C'})|B'C|$. Hence, we obtain,

$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = \frac{1}{1 + \frac{m_{C'}}{m_B}}\left(1 + \frac{m_{A'}}{m_B}\right)\frac{m_B + m_{C'}}{m_B + m_{A'}} = 1$$

**Theorem (Menelaus).** Points $A'$, $B'$ and $C'$ on the sides, or on the lines that suitably extend the sides $BC$, $AC$, and $AB$, of triangle ABC, are collinear (belong to the same line) if and only if,

$$\frac{|A'B|}{|A'C|}\frac{|B'C|}{|B'A|}\frac{|C'A|}{|C'B|} = 1$$

Menelaus's theorem provides a criterion for collinearity, just as Ceva's theorem provides a criterion for concurrence.

**Proof (similarity).** The statement could be proven with, or without using the method of point masses.

First, assume the points are collinear and consider rectangular triangles obtained by drawing perpendiculars onto the line A'B'. Using their similarity, one has
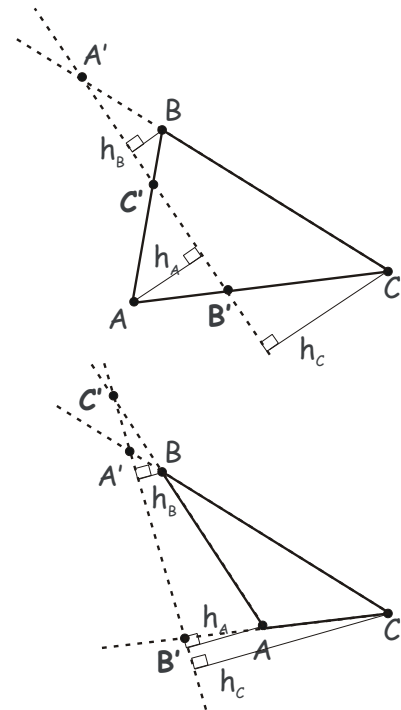
$$\frac{|A'B|}{|A'C|} = \frac{h_B}{h_C}, \frac{|B'C|}{|B'A|} = \frac{h_C}{h_A}, \frac{|C'A|}{|C'B|} = \frac{h_A}{h_B}$$

Wherefrom the statement of the theorem is obtained by multiplication (Coxeter & Greitzer).

**Proof (point masses).** Alternatively, let us load points $A$, $A'$ and $C$ in the upper Figure with the point masses $m_1$, $m_2$ and $m_3$, respectively. We select $m_1$, $m_2$ and $m_3$ such that $B'$ is the COM of $m_1(A)$ and $m_3(C)$, and $B$ is the COM of $m_2(A')$ and $m_3(C)$. The COM of all 3 masses belongs to both segments $AB$ and $A'B'$, which means that it is at point $C'$. Then,

$$\frac{|A'B|}{|A'C|} = \frac{m_3}{m_2 + m_3}, \frac{|B'C|}{|B'A|} = \frac{m_1}{m_3}, \frac{|C'A|}{|C'B|} = \frac{m_2 + m_3}{m_1}$$

Wherefrom the Menelaus theorem is obtained by multiplication. The case shown in the lower figure is considered in a similar way.