Homework for December 4, 2022.

Algebra.

Review the classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Rewrite the following properties of set algebra and partial ordering operations on sets in the form of logical propositions, following the first example.
    a. $[\mathrm{A}\cdot(\mathrm{B}+\mathrm{C}) = \mathrm{A}\cdot\mathrm{B}+\mathrm{A}\cdot\mathrm{C}] \Leftrightarrow \left[(\mathrm{x}\in\mathrm{A})\wedge\left((\mathrm{x}\in\mathrm{B})\vee(\mathrm{x}\in\mathrm{C})\right)\right] = \left[\left((\mathrm{x}\in\mathrm{A})\wedge(\mathrm{x}\in\mathrm{B})\right)\vee\left((\mathrm{x}\in\mathrm{A})\wedge(\mathrm{x}\in\mathrm{C})\right)\right]$
    b. $A+(B\cdot C) = (A+B)\cdot(A+C)$
    c. $(A\subset B)\Leftrightarrow A+B=B$
    d. $(A\subset B)\Leftrightarrow A\cdot B=A$
    e. $(A+B)'=A'\cdot B'$
    f. $(A\cdot B)'=A'+B'$
    g. $(A\subset B)\Leftrightarrow(B'\subset A')$
    h. $(A+B)'=A'\cdot B'$
    i. $(A'+B')'+(A'+B)'=A$
2. Using definitions from the classwork handout, devise logical arguments proving each of the following properties of algebra and partial ordering operations on sets and draw Venn diagrams where possible (hint: use problem #1).
    a. $\mathrm{A}\cdot(\mathrm{B}+\mathrm{C}) = \mathrm{A}\cdot\mathrm{B}+\mathrm{A}\cdot\mathrm{C}$
    b. $A+(B\cdot C) = (A+B)\cdot(A+C)$
    c. $(A\subset B)\Leftrightarrow A+B=B$
    d. $(A\subset B)\Leftrightarrow A\cdot B=A$
    e. $(A+B)'=A'\cdot B'$
    f. $(A\cdot B)'=A'+B'$
    g. $(A\subset B)\Leftrightarrow(B'\subset A')$
    h. $(A+B)'=A'\cdot B'$
    i. $(A'+B')'+(A'+B)'=A$
3. Verify that a set of eight numbers, {1,2,3,5,6,10,15,30}, where addition is identified with obtaining the least common multiple,

$$m+n\equiv LCM(n,m)$$

multiplication with the greatest common divisor,

$$m \cdot n \equiv GCD(n, m)$$

$m \subset n$ to mean "m is a factor of n",

$$m \subset n \equiv \left(n = 0 mod(m)\right)$$

and

$$n' \equiv 30/n$$

satisfies all laws of the set algebra.

4. For a set A, define the characteristic function χ_A as follows,

$$\chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x \ \notin A \end{cases}$$

Show that χ_A has following properties

$$\chi_A = 1 - \chi_{A'}$$

$$\chi_{A\cap B} = \chi_A \chi_B$$

$$\chi_{A\cup B} = 1 - \chi_{A'\cap B'} = 1 - \chi_{A'} \chi_{B'} = 1 - (1 - \chi_A)(1 - \chi_B)$$

$$= \chi_A + \chi_B - \chi_A \chi_B$$

Write formulas for $\chi_{A\cup B\cup C}, \chi_{A\cup B\cup C\cup D}$.

5. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

6. Using the continued fraction representation, find rational number, r, approximating $\sqrt{2}$ to the absolute accuracy of 0.0001.

7. Consider the values of the following expression, y, for different x. How does it depend on x when n becomes larger and larger?

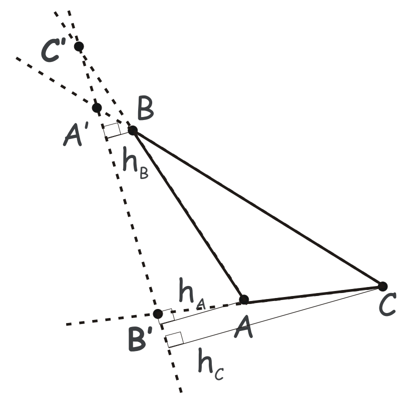
$$n\ fractions \left\{ y = 3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \cdots}}} \quad \cdots - \frac{2}{3 - x} \right. .$$

Geometry.

Review the last classwork handout on solving problems using mass points and the center of mass. Solve the unsolved problems from previous homeworks. Try solving the following problems (skip the ones you have already solved).

Problems.

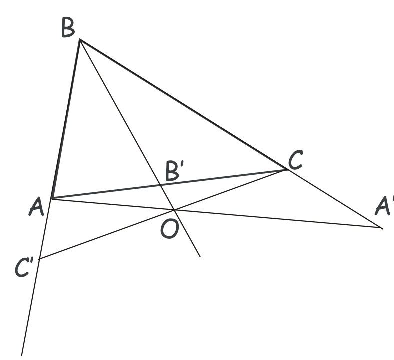
1. Prove Menelaus theorem for the configuration shown on the right using mass points. Menelaus theorem states,



Points C', A' and B', which belong to the lines containing the sides AB, BC and CA, respectively, of triangle ABC are collinear if and only if,

$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = 1$$

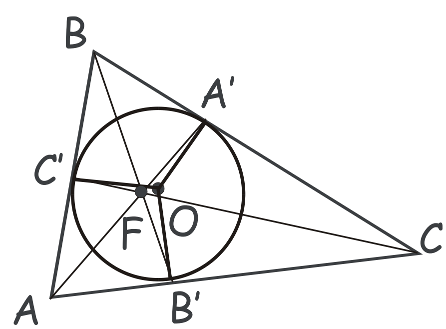
2. Prove the extended Ceva theorem (i) using mass points and the center of mass and (ii) using the similarity of triangles. Extended Ceva theorem states,



Segments (Cevians) connecting vertices A, B and C, with points A', B' and C' on the sides, or on the lines that suitably extend the sides BC, AC, and AB, of triangle ABC, are concurrent if and only if,

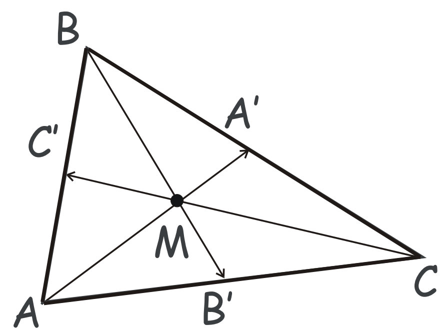
$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = 1$$

3. In a triangle ABC, A', B' and C' are the tangent points of the inscribed circle and the sides BC, AC, and AB, respectively (see Figure). Prove that cevians AA', BB' and CC' are concurrent (their common point F is called the Gergonne point).



4. Points P and Q on the lateral sides AB and BC of an isosceles triangle ABC divide these sides into segments whose lengths have ratios $|AP|:|PB| = n$, and $|BQ|:|QC| = n$. Segment PQ crosses altitude BB' at point M. What is the ratio $|BM|:|MB'|$ of two segments into which PQ divides the altitude BB'?

5. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).



6. Prove that in a right triangle, each side of the right angle is the geometric mean between the whole hypotenuse and its projection onto the hypotenuse. That is, if BD is the altitude from the vertex of the right angle, ABC, onto the hypotenuse, AC, then $|AB|^2 = |AC||AD|$.

7. Prove that three medians in a triangle divide it into six smaller triangles of equal area.