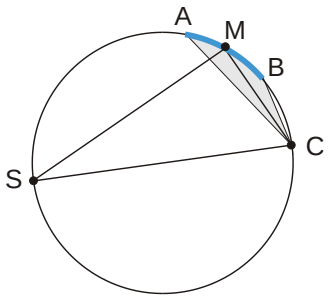
Homework for December 18, 2022.

Geometry.

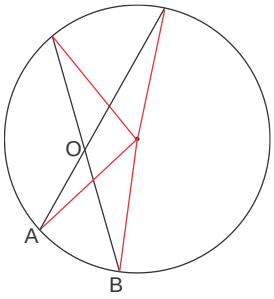
Review the previous classwork notes. Solve the following problems, including problems from the last homework (if you have not solved them yet).

Problems.

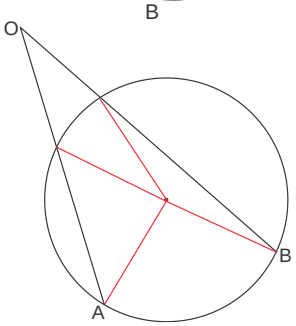
1. A **Rowland focusing** mirror is a device which focuses light of a certain color from the point source S onto a point, C, at sample. The mirror has the shape of a circular arc AB of 40 cm length. It is positioned so that its center, M, is at a distance 4 m from the point source S and at a distance 2 m from the sample C, $|SM| = 4$ m, $|MC| = 2$ m. The light ray of the color of interest is reflected so that it forms a 90° angle with the incident ray (e.g. angle SMC in the figure on the right is 90°).



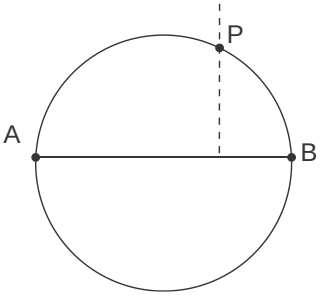
   a. What is the radius of the Rowland circle?
   b. What is the angular size of the light beam illuminating the sample (shaded angle ACB in the figure)? Does it depend on the position of sample, C?



2. Prove that an angle whose vertex lies inside a disk is measured by a semi-sum of the two arcs, one of which is intercepted by this angle, and the other by the angle vertical to it.



3. Prove that an angle whose vertex lies outside a disk and whose sides intersect the circle, is measured by a semi-difference the two intercepted arcs.
4. Given a circle and a diameter drawn of that circle, using only a straightedge, draw a perpendicular to that diameter passing through (i) point P on the circle; (ii) point P outside the circle (iii) point P inside the circle.



Algebra.

Review the last classwork handout. Review and solve the classwork exercises which were not solved and unsolved problems from the previous homeworks. Solve the following problems (skip the ones that you have already solved).

1. Prove the following properties of the Cartesian product,
   a. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   c. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
2. Find the Cartesian product, $A \times B$, of the following sets,
   a. $A = \{a, b\}, B = \{\uparrow, \downarrow\}$
   b. $A = \{June, July, August\}, B = \{1,15\}$
   c. $A = \emptyset, B = \{1,2,3,4,5,6,7,8,9\}$
3. Describe the set of points determined by the Cartesian product, $A \times B$, of the following sets (illustrate schematically on a graph),
   a. $A = [0,1], B = [0,1]$ (two segments from 0 to 1)
   b. $A = [-1,1], B = (-\infty, \infty)$
   c. $A = (-\infty, 0], B = [0, \infty)$
   d. $A = (-\infty, \infty), B = (-\infty, \infty)$
   e. $A = [0,1), B = \mathbb{Z}$ (set of all integers)
4. Propose 3 meaningful examples of a Cartesian product of two sets.
5. $n_A = |A|$ is the number of elements in a set A.
   a. What is the number of elements in a set $A \times A$
   b. What is the number of elements in a set $A \times (A \times A)$

<u>Recap problems from previous homeworks: solve the ones you have not yet solved</u>

6. Using the method of mathematical induction, prove the following equality,

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1$$

7. Put the sign $<$, $>$, or $=$, in place of … below,

$$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

8. Find the following sum.

$$\left(2+\frac{1}{2}\right)^2+\left(4+\frac{1}{4}\right)^2+\cdots+\left(2^n+\frac{1}{2^n}\right)^2$$

9. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?
10. Solve the following equation,

$\frac{x-1}{x}+\frac{x-2}{x}+\frac{x-3}{x}+\cdots+\frac{1}{x}=3$, where x is a positive integer.

11. Find the following sum,
    a. $1+2\cdot 3+3\cdot 7+\cdots+n\cdot(2^n-1)$
    b. $1\cdot 3+3\cdot 9+5\cdot 27+\cdots+(2n-1)\cdot 3^n$
12. Numbers $a_1, a_2, \ldots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

$$S_n = a_1 a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)$$

13. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{1}{3-\sqrt{3}}+\frac{1}{6}+\cdots$$

14. What is the maximum value of the expression, $(1+x)^{36}+(1-x)^{36}$ in the interval $|x|\leq 1$?
15. Find the coefficient multiplying x^9 after all parenthesis are expanded in the expression, $(1+x)^9+(1+x)^{10}+\cdots+(1+x)^{19}$.