Homework for January 8, 2023.

Algebra.

Review the previous classwork handouts. Solve the remaining problems from the previous homework assignments and classwork exercises. Specifically, complete the assignment on inclusion-exclusion principle. Solve the following problems.

1. Construct bijections between the following sets:
   a. (subsets of the set $\{1, \ldots, n\}$) $\leftrightarrow$ (sequences of zeros and ones of length n)
   b. (5-element subsets of $\{1, \ldots, 15\}$) $\leftrightarrow$ (10-element subsets of $\{1, \ldots, 15\}$)
   c. [set of all ways to put 10 books on two shelves (order on each shelf matters)] $\leftrightarrow$ (set of all ways of writing numbers 1, 2, . . . , 11 in some order) [Hint: use numbers 1. . . 10 for books and 11 to indicate where one shelf ends and the other begins.]
   d. (all integer numbers) $\leftrightarrow$ (all even integer numbers)
   e. (all positive integer numbers) $\leftrightarrow$ (all integer numbers)
   f. (interval $(0,1)$) $\leftrightarrow$ (interval $(0,5)$)
   g. (interval $(0,1)$) $\leftrightarrow$ (halfline $(1, \infty)$) [Hint: try $1/x$.]
   h. (interval $(0,1)$) $\leftrightarrow$ (halfline $(0, \infty)$)
   i. (all positive integer numbers) $\leftrightarrow$ (all integer numbers)
2. Let A be a finite set, with 10 elements. How many bijections f: $A \to A$ are there? What if A has n elements?
3. Let $f: \mathbb{Z} \xrightarrow{\text{f}} \mathbb{Z}$ be given by $f(n) = n^2$. Is this function injective? surjective?
4. Hotel Infinity is a fictional hotel with infinitely many rooms, numbered 1, 2, 3, ... Each hotel room is single occupancy: only one guest can stay there at any time.
   a. At some moment, Hotel Infinity is full: all rooms are occupied. Yet, when 2 more guests arrive, the hotel manager says he can give rooms to them, by moving some of the current guests around. Can you show how? (Hint: Construct a bijection between the set $\{-1, 0, 1, 2, \ldots\}$ and the set of natural numbers, $\mathbb{N}$).

b. At some moment, Hotel Infinity is full: all rooms are occupied. Still, the management decides to close half of the rooms — all rooms with odd numbers — for renovation. They claim they can house all their guests in the remaining rooms. Can you show how? (Hint: Construct a bijection between the set of all even positive integers $\{2,4,6,...\}$ and $\mathbb{N}$).

c. Next to Hotel infinity, a competitor has built Hotel Infinity 2, with infinitely many rooms numbered by all integers: $...,-2,-1,0,1,2,...$. Yet, the management of the original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 2 in Hotel Infinity. Could you show how? (Hint: Construct a bijection between the set of all integer numbers $\{...,-2, -1,0,1,2,...\}$ and $\mathbb{N}$).

Bonus recap problems

5. Find the value of the continued fraction given by

$$\{1,2,3,3,3,...\} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}}}$$

6. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

7. Using the recurrence relation obtained by solving the old hats problem, derive the formula for a derangement probability,

$$p_n \equiv \frac{!n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

8. Using the inclusion-exclusion principle, find how many natural numbers $n < 1000$ are divisible by 5, 7, 11, or 13.

9. How many passwords of at least 8 characters can one compose using lower- and upper-case letters and numbers 0 to 9?

10. * If 9 dies are rolled, what is the probability that all 6 numbers appear?

11. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?

Geometry.

Review the last classwork handout on inscribed angles and quadrilaterals. Go over the proofs of Ptolemy's and Euclid's theorems. Try solving the following problems including the unsolved problems from previous homeworks.

Problems.

1. Write the proof of the Euclid theorem, which states the following. If two chords AD and BC intersect at a point P' outside the circle, then

$$|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$$

where $|PT|$ is a segment tangent to the circle (see Figure).

2. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.
   a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?
   b. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?

3. Using the Ptolemy's theorem, prove the following:
   a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
   b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.
   c. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle?

4. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.
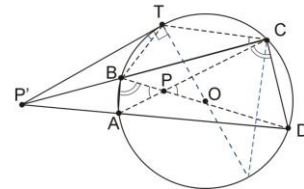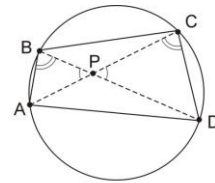
5. Consider all triangles with a given base and given altitude corresponding to this base. Prove that among all these triangles the isosceles triangle has the biggest angle opposite to the base.

6. Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.

7. *In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.

8. * Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,
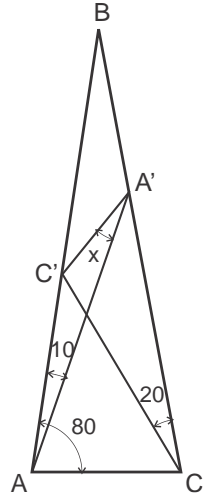
$$|AC| \cdot |BD| \le |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).