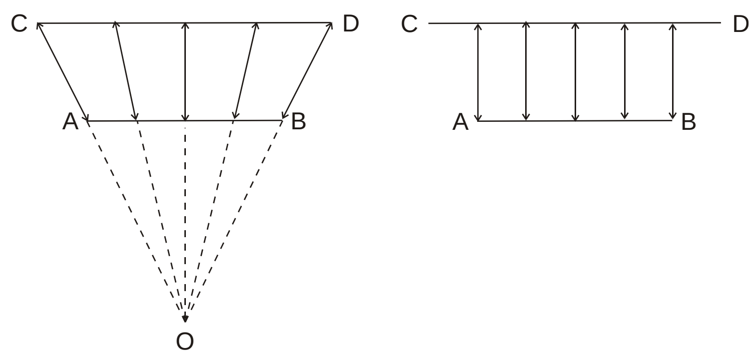
January 8, 2023

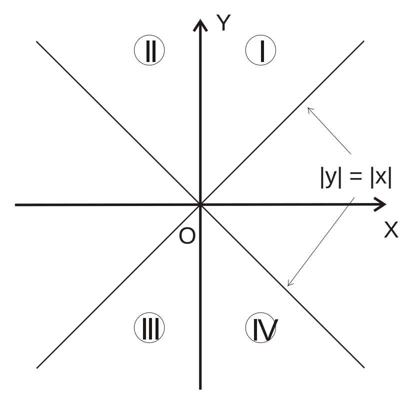
Geometry.

The method of coordinates.

In introducing coordinates, we set up a correspondence between numbers and points on a straight line. The following property is satisfied: to each point on the line there corresponds one and only one number, and to each number there corresponds one and only one point.



A correspondence between two sets where for each element of the first set there is one element in the second set, and each element in the second set corresponds to some element of the first set is called a **one-to-one correspondence**. Can there be a one-to-one correspondence between two line segments of unequal length (see Figure above)?



Which set of points is defined by the following relation (draw it on the coordinate plane)?

a. $|x| = |y|$

b. $|x| + x = |y| + y$

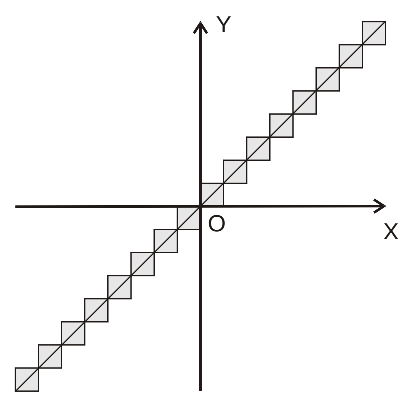
c. $|x|/x = |y|/y$

d. $x^2 - y^2 < 0$

e. $x^2 + y^2 > 1$

f. $x^2 + 8x = 9 - y^2$

g. $[y] = [x]$

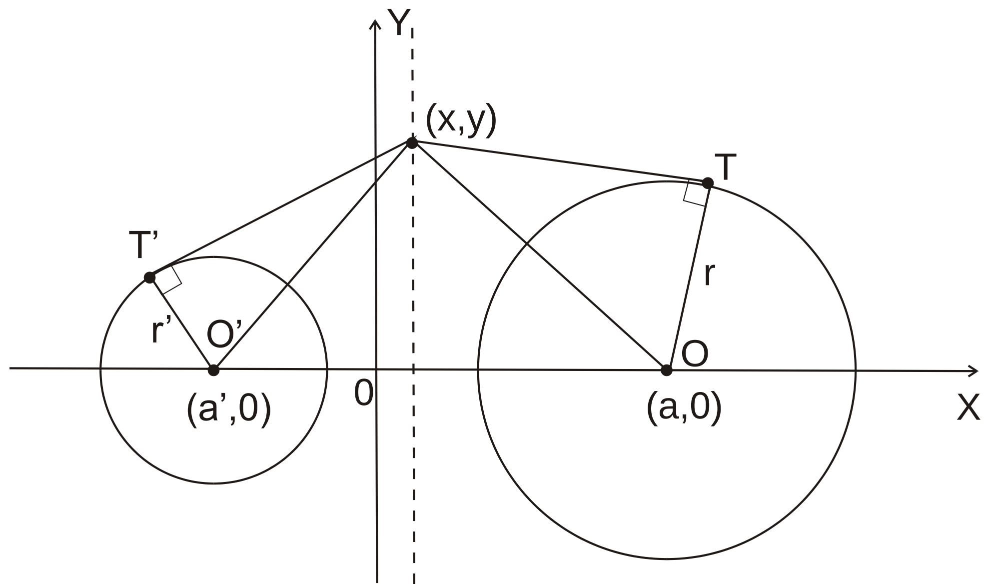


h. $\{y\} = \{x\}$

The radical axis of two circles.

Recall that for any circle of radius R and any point P distant d from the center, the quantity $d^2 - R^2$ is called the power of P with respect to the circle.

Consider the locus of all points whose powers with respect to two non-concentric circles are equal. These points form a straight line perpendicular to the line of centers of the two circles. This line is called the radical axis of two circles.



The easiest proof is by writing the relation for coordinates (x, y) of such points (Coxeter, Gretzer, pp. 31-33),

$(x - a')^2 + y^2 - r'^2 = (x - a)^2 + y^2 - r^2$

$(a' - a)(2x - a - a') = r^2 - r'^2.$

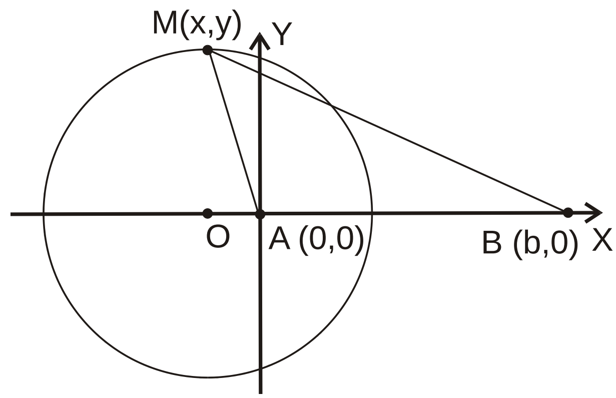
This defines line $x = const$, which is perpendicular to the X-axis.

Exercise. Given two points A and B, prove that the locus of points M, such that $|MA|^2 - |MB|^2 = kS_{MAB}$, is a pair of straight lines.

The circle of Apollonius.

Problem. Given two points A and B in the plane, find the locus of points M whose distance from A is k times as great as from B.

Solution. Let us choose a system of coordinates on the plane such that point A is at the origin, (0,0), and point B is on the X-axis, and has coordinates $(b, 0)$. Let point $M(x, y)$ satisfy the condition of the problem, $|AM| = k|BM|$, or,



$$\sqrt{x^2 + y^2} = k\sqrt{(x-b)^2 + y^2} \Leftrightarrow$$

$$x^2 + y^2 = k^2(x-b)^2 + k^2y^2 \Leftrightarrow (1-k^2)x^2 + 2k^2bx + (1-k^2)y^2 = k^2b^2 \Leftrightarrow$$

$$x^2 + \frac{2k^2bx}{(1-k^2)} + y^2 = \frac{k^2b^2}{(1-k^2)} \Leftrightarrow$$

$$\left(x + \frac{k^2b}{(1-k^2)}\right)^2 + y^2 = \frac{k^2b^2}{(1-k^2)} + \left(\frac{k^2b}{(1-k^2)}\right)^2 = \frac{k^2b^2}{(1-k^2)^2}$$

So the sought locus of points is a circle with the center at $\left(-\frac{k^2b}{(1-k^2)}, 0\right)$ and with the radius, $R = \frac{kb}{(1-k^2)}$. Note, that in the above solution we have assumed $k < 1$. Solution for $k > 1$ is similar, but because $1 - k^2$ is negative, its sign should be accounted for when taking the square root, so $R = \frac{kb}{(k^2-1)}$. In the special case, $k = 1$, the answer is obvious: it is a line, perpendicular to the segment AB and passing through its center (perpendicular bisector).

Exercise. Given the triangle ABC with sides $|BC| = a$, $|AC| = b$ and $|AB| = c$, find the center and the radius of the circle circumscribed around this triangle.

Apollonius' problem (the tangent circles construction).

The Problem of Apollonius is the subject of two lost books, The Tangencies, by Apollonius of Perga (c. 262-190 BC). Apollonius was a Greek geometer and astronomer noted for his writings on conic sections. In his surviving work Conics, Apollonius who gave the ellipse, the parabola, and the hyperbola the names by which we know them. We know of the Problem of Apollonius though the writing of Pappus of Alexandria (c. 290-350) - a famous geometer in his own right. Here is a quote from the fragment in [Greek Mathematical Works, 342-343]:

Given three entities, of which any one may be a point or a straight line or a circle, to draw a circle which shall pass through each of the given points, so far as it is points which are given, or to touch each of the given lines.

In this problem, according to the number of like or unlike entities in the hypotheses, there are bound to be, when the problem is subdivided, ten enunciations. For the number of different ways in which three entities can be taken out of three unlike sets is ten. The most difficult and famous case is when the entities are circles.

Problem. Given three arbitrary circles in the plane, find a fourth circle tangent to the first three.

Solution. Let the centers of the three given circles be (x_1, y_1), (x_2, y_2), (x_3, y_3), and their radii r_1, r_2 and r_3, respectively. Let the fourth circle tangent tot these three have center (x, y) and radius r. Then, the tangency conditions can be written as,

$$(x - x_1)^2 + (y - y_1)^2 = (r \pm r_1)^2,$$

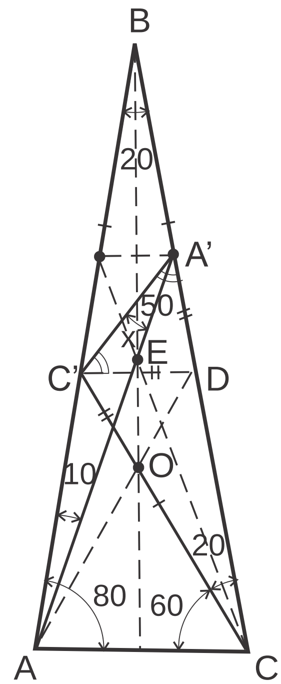
$$(x - x_2)^2 + (y - y_2)^2 = (r \pm r_2)^2,$$

$$(x - x_3)^2 + (y - y_3)^2 = (r \pm r_3)^2.$$

Thus we have obtained three quadratic equations for the three unknown variables, (x, y, r). Because quadratic terms in all three equations are identical, they can be reduced to linear equations by mutual subtraction and this way easily solved.

<u>Solutions to some homework problems</u>.

1. **Problem**. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.

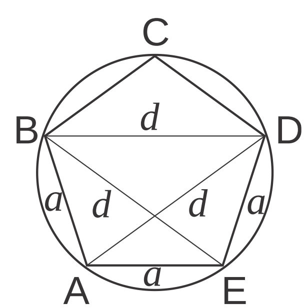


   Solution. Making supplementary constructs shown in the figure, we see that $\Delta DOC'$ is equilateral (all angles are $60°$), while $|OD| = |C'O| = |A'D|$ as corresponding elements in congruent triangles, $\Delta BOC' \cong \Delta BOD \cong \Delta AA'D$. Hence, $\Delta A'C'D$ is isosceles ns $\angle A'C'D = \angle CA'D = 50°$, wherefrom, $x = \angle CA'D - \angle AA'D = 50° - 30° = 20°$.

2. **Problem**. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).

   Solution. Load vertices A, B and C with masses $m_A = 1, m_B = p$, and $m_C = q$, respectively. This makes point C' is on the side AB center of mass for $m_A = 1$ and $m_B = p$ and point B' is on the side AC center of mass for $m_A = 1$ and $m_C = q$. Point M is then the center of mass for all three masses. Moving masses $m_B = p$, and $m_C = q$ to their center of mass A' is on the side BC and using the lever rule, we obtain, $\frac{AM}{MA'} = p + q$.

3. **Problem**. Using the Ptolemy's theorem, prove the following:
   a. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.



   Solution. Consider the circumscribed circle (why can be circle circumscribed around a regular pentagon?). A diagonal divides pentagon into an isosceles triangle and a quadrilateral (trapezoid) inscribed in the same circle. The sides of the trapezoid are a, a, a, and d, where a is the side of the pentagon and d its diagonal. Applying the

Ptolemy's theorem we obtain, $a \cdot a + a \cdot d = d \cdot d$, or, $\left(\frac{d}{a}\right)^2 - \left(\frac{d}{a}\right) - 1 = 0$, which is the equation for the golden ratio, $\frac{d}{a} = \frac{1+\sqrt{5}}{2}$.