Homework for February 5, 2023

Algebra.

Review the last algebra classwork handouts. Solve the unsolved problems from the previous homework repeated below.

1. Assume that the set of rational numbers $\mathbb{Q}$ is divided into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$: $\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b$.
    a. Prove that if $\mathbb{Q}_>$ contains the smallest element, $\exists b_0 \in \mathbb{Q}_>, \forall b \in \mathbb{Q}_>, b_0 \leq b$, then $\mathbb{Q}_<$ does not contain the largest element
    b. Prove that if $\mathbb{Q}_<$ contains the largest element, $\exists a_0 \in \mathbb{Q}_<, \forall a \in \mathbb{Q}_<, a \leq a_0$, then $\mathbb{Q}_>$ does not contain the smallest element
    c. Present an example of such a partition, where neither $\mathbb{Q}_>$ contains the smallest element, nor $\mathbb{Q}_<$ contains the largest element
2. Prove the following properties of countable sets. For any two countable sets, A, B,
    a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
    b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
    c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$

3. Let W be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]
4. Compare the following real numbers (are they equal? which is larger?)
    a. 1.33333… = 1.(3) and 4/3
    b. 0.09999… = 0.0(9) and 1/10
    c. 99.9999… = 99.(9) and 100
    d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$
5. Simplify the following real numbers. Are these numbers rational? (hint: you may use the formula for an infinite geometric series).

a. 1/1.1111…=1/1.1(1)
b. 2/1.2323…=2/1.23(23)
c. 3/0.123123…=3/0.123(123)

6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. 1/8
   b. 2/7
   c. 0.1
   d. $0.33333\ldots = 0.(3)$
   e. $0.13333\ldots = 0.1(3)$
7. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\, a, b \in \mathbb{R}$, one and only one of the following relations holds
   - $a = b$
   - $a < b$
   - $a > b$
2. $\forall\, a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

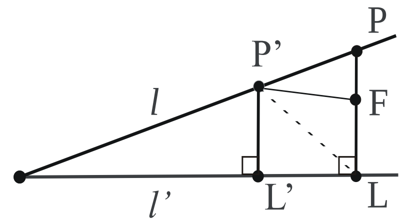
- $\forall\, a, b \in \mathbb{R}, a + b = b + a$
- $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\, a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\, a \in \mathbb{R}, \exists - a \in \mathbb{R}, a + (-a) = 0$
- $\forall\, a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\, a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

Geometry.

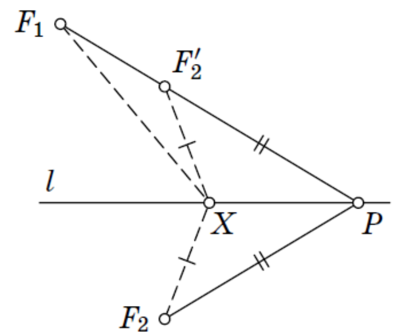
Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework.

Problems.

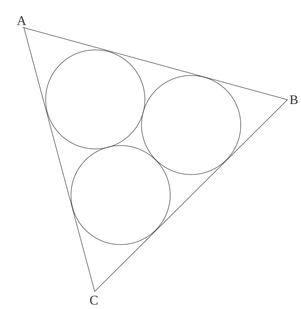
1. Given two lines, l and l', and a point F not on any of those lines, find point P on l such that the (signed) difference of distances from it to l' and F, $|P'L'| - |P'F|$, is maximal. As seen in the figure, for any P' on l the distance to l', $|P'L'| \leq |P'L| \leq |P'F| + |FL|$, where $|FL|$ is the distance from F to l'. Hence, $|P'L'| - |P'F| \leq |FL|$, and the difference is largest ($= |FL|$) when point P belongs to the perpendicular FL from point F to l'.



2. Given line l and points F_1 and F_2 lying on different sides of it, find point P on the line l such that the absolute value of the difference in distances from P to points F_1 and F_2 is maximal. As above, let $F_2{}'$ be the reflection of F_2 in l. Then for any point X on l, $|XF_2| - |XF_1'| \leq |F_1F_2'|$.



3. Find the (x, y) coordinates of the common (intersection) point of the two lines, one passing through the origin at 45 degrees to the X-axis, and the other passing through the point (1,0) at 60 degrees to it.
4. Find the (x, y) coordinates of the common (intersection) points of the parabola $y = x^2$ and of the ellipse centered at the origin and with major axis along the Y-axis whose length equals 2, and the minor axis along the X-axis whose length equals 1.
5. (Skanavi 10.122) Find the locus of the midpoints of all chords of a given circle with the center O, which intersect given chord AB of this circle.



6. Three circles of radius r touch each other. Find the area of the triangle ABC formed by tangents to pairs of circles (see figure).