Homework for February 12, 2023

Geometry.

Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework (some are repeated below – skip the ones you have already done).

Problems.

1. Review derivation of the equation describing an ellipse and derive in a similar way,
   a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
   b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
2. Find (describe) set of all points formed by the centers of the circles that are tangent to a given circle of radius r and a line at a distance $d > r$ from its center, O.
3. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.
4. Find the equation of the locus of points equidistant from two lines, $y = ax + b$ and $y = mx + n$, where a, b, m, n are real numbers.
5. Find the distance between the nearest points of the circles,
   a. $(x-2)^2 + y^2 = 4$ and $x^2 + (y-1)^2 = 9$
   b. $(x+3)^2 + y^2 = 4$ and $x^2 + (y-4)^2 = 9$
   c. $(x-2)^2 + (y+1)^2 = 4$ and $(x+1)^2 + (y-3)^2 = 5$
   d. $(x-a)^2 + y^2 = r_1^2$ and $x^2 + (y-b)^2 = r_2^2$

Algebra.

Review the classwork handout and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. 1/5
   b. 1/9
   c. 1/17
2. Show that the only possible remainders of division of the square of a natural number, n^2, by 3 are 0 and 1. What are the possible remainders of division of the square of a natural number, n^2, by 7?
3. Let W be the set of all "words" that can be written using the alphabet consisting of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]
4. If 9 dies are rolled, what is the probability that all 6 numbers appear? Using the solution of this problem, prove the following formula,

$$6! = 6^6 - 6 \cdot 5^6 + 15 \cdot 4^6 - 20 \cdot 3^6 + 15 \cdot 2^6 - 6$$

5. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?
6. Represent $\sqrt{2}$ (and $\sqrt{p}$ for any rational p) by using the continuous fraction,

$$\sqrt{2} = a + \cfrac{c}{b + \cfrac{c}{b + \cfrac{c}{b + \cdots}}}$$