March 5, 2023 Math 9

Trigonometry.

Trigonometry homework review.

The following trigonometric formulas will be useful for solving the homework.

1. Products of sine and cosine

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$$

$$\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta)-\sin(\alpha-\beta))$$

2. Sums of sine and cosine

$$\cos(\alpha)+\cos(\beta) = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos(\alpha)-\cos(\beta) = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\sin(\alpha)+\sin(\beta) = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin(\alpha)-\sin(\beta) = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

3. Sine and cosine of double and triple angle

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2\cot\alpha}{1+\cot^2\alpha}$$

$$\cos 2\alpha = \cos^2\alpha-\sin^2\alpha = 2\cos^2\alpha-1 = 1-2\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{\cot^2\alpha-1}{\cot^2\alpha+1}$$

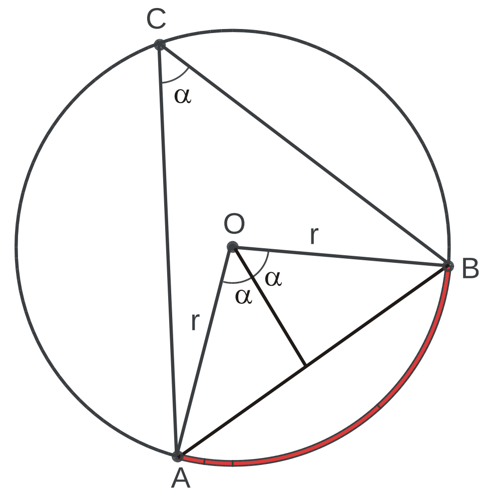
$$\sin 3\alpha = 3\sin\alpha-4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha-3\cos\alpha$$

Solutions to selected homework problems

Problems.

1. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.



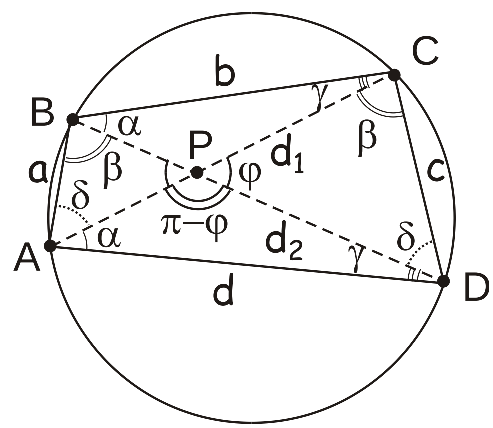
2. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity (see Figure):

   $\sin(\alpha+\beta)\sin(\beta+\gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta,$

   if $\alpha+\beta+\gamma+\delta=\pi$.

3. Prove the Ptolemy identity in Problem 2 using the addition formulas for sine and cosine.

Solutions.



1. Consider the figure on the right,

$|AB| = 2r\sin\alpha = \sin\alpha$, if $d = 2r = 1$.

2. According to Ptolemy's theorem for a quadrilateral inscribed in a circle,

$d_1 d_2 = ac + bd.$

Applying this for the circle of the unit diameter and using the result of the previous problem, we obtain,

$\sin(\alpha+\beta)\sin(\beta+\gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta$, where $\alpha+\beta$ and $\gamma+\delta$ are the opposite angles of an inscribed quadrilateral (and so are $\alpha+\delta$ and $\beta+\gamma$), and therefore $\alpha+\beta+\gamma+\delta=\pi$.

3. Using the multiplication formulas for sines we obtain,

$$\sin\alpha\sin\gamma + \sin\beta\sin\delta = \frac{1}{2}[\cos(\alpha-\gamma) - \cos(\alpha+\gamma) + \cos(\beta-\delta) - \cos(\beta+\delta)] = \frac{1}{2}\left[\cos(\alpha-\gamma) + \cos(\beta-\delta) - \left(\cos(\alpha+\gamma)+\cos(\beta+\delta)\right)\right]$$
$$= \frac{1}{2}\left[2\cos\left(\frac{\alpha-\gamma+\beta-\delta}{2}\right)\cos\left(\frac{\alpha-\gamma-\beta+\delta}{2}\right)\right] = \cos\left(\frac{2\alpha+2\beta-\pi}{2}\right)\cos\left(\frac{\pi-2\gamma-2\beta}{2}\right) = \sin(\alpha+\beta)\sin(\beta+\gamma).$$

4. Using the Sine and the Cosine theorems, prove the Hero's formula for the area of a triangle,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter.

Solution. The area of a triangle ABC is $S_{\Delta ABC} = \frac{1}{2}ab\sin\gamma$, so

$$\sin^2\gamma = \frac{4{S_{\Delta ABC}}^2}{a^2b^2}$$

From the Law of cosines, we have

$$\cos^2\gamma = \left(\frac{a^2+b^2-c^2}{2ab}\right)^2$$

Adding the two expressions, we obtain, $1 = \frac{4{S_{\Delta ABC}}^2}{a^2b^2} + \frac{(a^2+b^2-c^2)^2}{4a^2b^2}$, or,

$16{S_{\Delta ABC}}^2 = 4a^2b^2 - (a^2+b^2-c^2)^2 = (2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2) = ((a+b)^2-c^2)(c^2-(a-b)^2)$$=(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$, or,

$${S_{\Delta ABC}}^2 = p(p-a)(p-b)(p-c)$$

5. Show that

a. $\cos^2\alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \cos^2\alpha + \left(-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right)^2 + \left(-\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right)^2 = \cos^2\alpha + 2\left(\frac{1}{2}\cos\alpha\right)^2 + 2\left(\frac{\sqrt{3}}{2}\sin\alpha\right)^2 = \frac{3}{2}\cos^2\alpha + \frac{3}{2}\sin^2\alpha = \frac{3}{2}$

b. $\sin\alpha + \sin\left(\frac{2\pi}{3}+\alpha\right) + \sin\left(\frac{4\pi}{3}+\alpha\right) = \sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha = 0$

c. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = \frac{3\sin x - 4\sin^3 x}{\sin x} - \frac{4\cos^3 x - 3\cos x}{\cos x} = 6 - 4\sin^2 x - 4\cos^2 x = 2$

6. Without using calculator, find:

a. $\sin 75° = \sin(90° - 15°) = \cos 15° = \cos\frac{30°}{2} = \sqrt{\frac{1}{2}(1 + \cos 30°)} = \sqrt{\frac{2+\sqrt{3}}{4}}$

b. $\cos 75° = \cos(90° - 15°) = \sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1}{2}(1 - \cos 30°)} = \sqrt{\frac{2-\sqrt{3}}{4}}$

c. $\sin\frac{\pi}{8} = \sin\frac{1}{2}\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\frac{\pi}{4}\right)} = \sqrt{\frac{2-\sqrt{2}}{4}}$

d. $\cos\frac{\pi}{8} = \cos\frac{1}{2}\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right)} = \sqrt{\frac{2+\sqrt{2}}{4}}$

e. $\sin\frac{\pi}{16} = \sin\frac{1}{2}\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\frac{\pi}{8}\right)} = \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}}$

f. $\cos\frac{\pi}{16} = \cos\frac{1}{2}\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{8}\right)} = \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}}$

g. $\cos\frac{\pi}{2^{n+1}} = \cos\frac{1}{2}\left(\frac{\pi}{2^n}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{2^n}\right)} = \sqrt{\frac{2+\sqrt{2+\sqrt{2+\cdots}}}{4}}$

Trigonometric series.

Problem. Using the trigonometric formulas that we have previously derived, find the sum of the following trigonometric series,

$$S_N = \sin x + \sin 2x + \sin 3x + \sin 4x + \cdots + \sin Nx$$

Solution. Multiplying the sum with $2\sin\frac{x}{2}$ and using the formula for the product of two sines and then for the difference of two cosines we obtain,

$$2\sin\frac{x}{2}\cdot S_N = \cos\frac{x}{2} - \cos\frac{3x}{2} + \cos\frac{3x}{2} - \cos\frac{5x}{2} + \cdots + \cos\left(Nx - \frac{x}{2}\right) - \cos\left(Nx + \frac{x}{2}\right) = \cos\frac{x}{2} - \cos\left(Nx + \frac{x}{2}\right) = 2\sin Nx\sin(N+1)x$$

Consequently,

$$S_N = \frac{\sin Nx \sin(N+1)x}{\sin\frac{x}{2}}$$

Problem 2. Find the sum of the following series,

$$S = \cos x + \cos 2x + \cos 3x + \cos 4x + \cdots + \cos Nx$$

(hint: multiply the sum by $2\sin x/2$)

Solution 1: Easy way of summing the trigonometric series is by multiplying and dividing it with $\sin\frac{x}{2}$,

$S\,\frac{\sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}(\cos x+\cos 2x+\cdots+\cos nx)}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}\cos x+\sin\frac{x}{2}\cos 2x+\cdots+\sin\frac{x}{2}\cos nx}{\sin\frac{x}{2}} =$
$\frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\frac{3x}{2}-\sin\frac{3x}{2}+\sin\frac{5x}{2}-\sin\frac{5x}{2}+\cdots-\sin\left(n-\frac{1}{2}\right)x+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} = \frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} =$
$\frac{\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$

Solution 2. A different and perhaps easier way of summing the above trigonometric series is by adding the expression for S_1, or S_2, rearranged from back to front, to itself , as we did when summing the arithmetic series,

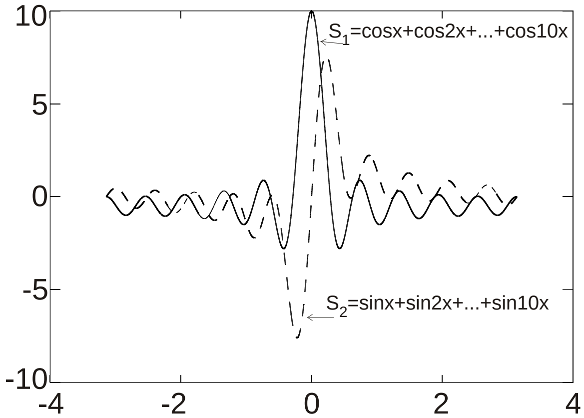
$$S_1 = \cos x + \cos 2x + \cdots + \cos nx$$

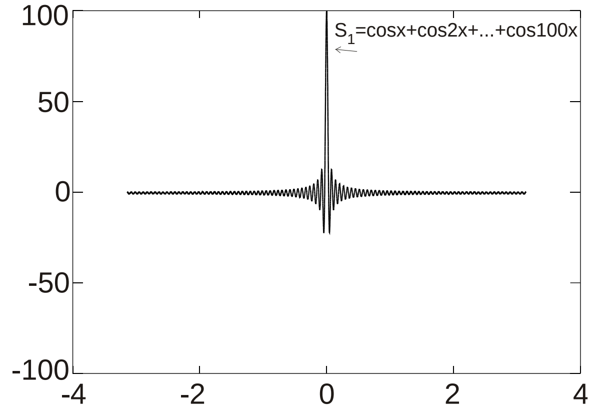
$$S_1 = \cos nx + \cos(n-1)x + \cdots + \cos x$$

Wherefrom,

$\mathrm{S_1} = \frac{1}{2}\left((\cos \mathrm{x} + \cos \mathrm{nx}) + (\cos 2\mathrm{x} + \cos(\mathrm{n}-1)\mathrm{x}) + \cdots + (\cos \mathrm{nx} + \cos \mathrm{x})\right) =$
$\cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\left(\cos(\mathrm{n}-1)\frac{\mathrm{x}}{2} + \cos(\mathrm{n}-3)\frac{\mathrm{x}}{2} + \cdots + \cos(\mathrm{n}-1)\frac{\mathrm{x}}{2}\right) =$
$\cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\frac{\left(\sin\frac{\mathrm{x}}{2}\cos(\mathrm{n}-1)\frac{\mathrm{x}}{2}+\sin\frac{\mathrm{x}}{2}\cos(\mathrm{n}-3)\frac{\mathrm{x}}{2}+\cdots+\sin\frac{\mathrm{x}}{2}\cos(\mathrm{n}-1)\frac{\mathrm{x}}{2}\right)}{\sin\frac{\mathrm{x}}{2}} =$
$\cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\frac{\frac{1}{2}\left(\sin\frac{\mathrm{nx}}{2}-\sin\frac{(\mathrm{n}-2)\mathrm{x}}{2}+\sin\frac{(\mathrm{n}-2)\mathrm{x}}{2}-\sin\frac{(\mathrm{n}-4)\mathrm{x}}{2}+\cdots+\sin\frac{\mathrm{nx}}{2}\right)}{\sin\frac{\mathrm{x}}{2}} = \cos\frac{(\mathrm{n}+1)\mathrm{x}}{2}\frac{\sin\frac{\mathrm{nx}}{2}}{\sin\frac{\mathrm{x}}{2}}.$

It is interesting to look at a function $S\,(x)$.





Behavior of $S_1(x)$ is intuitively clear. For $x = 0$, all terms in the sum are equal to 1, and the sum equals to the number of terms, $S_1(0) = n$, while for $x \neq 0$ it consists of many positive and negative terms, which tend to cancel each other.