Homework for April 16, 2023.

Algebra/Geometry. Complex numbers.

Please, complete the previous homework assignments. Review the classwork handout on complex numbers and complete the exercises. Solve the following problems.

Problems.

1. Find all complex numbers z such that:
    a. $z^2 = -i$
    b. $z^2 = -2 + 2i\sqrt{3}$
    c. $z^3 = i$

    Hint: write and solve equations for a, b in $z = a + bi$.

2. On the complex plane, plot all fifth order roots of 1 and all fifth order roots of -1.
3.
    a. Find all roots of the polynomial $z + z^2 + z^3 + \cdots + z^n$
    b. Without doing the long division, show that $1 + z + z^2 + \cdots + z^9$ is divisible by $1 + z + z^2 + z^3 + z^4$.

4. Find the roots of the following cubic equations by heuristic guess-and-check factorization,
    a. $z^3 - 7z + 6 = 0$
    b. $z^3 - 21z - 20 = 0$
    c. $z^3 - 3z = 0$
    d. $z^3 + 3z = 0$
    e. $z^3 - \frac{3}{4}z + \frac{1}{4} = 0$

5. Which transformation of the complex plane is defined by:
    a. $z \to iz$
    b. $z \to \left(\frac{1-i}{\sqrt{2}}\right) z$

c. $z \to \left(1 + i\sqrt{3}\right)z$

d. $z \to \frac{z}{1+i}$

e. $z \to \frac{z+\bar{z}}{2}$

f. $z \to 1 - 2i + z$

g. $z \to \frac{z}{|z|}$

h. $z \to i\bar{z}$

i. $z \to -\bar{z}$

6. Find the sum of the following trigonometric series using de Moivre formula:

$$S_1 = \cos x + \cos 2x + \cdots + \cos nx = ?$$

$$S_2 = \sin x + \sin 2x + \cdots + \sin nx = ?$$

Geometry. Vectors.

Please, complete problems from the previous homework assignment. Review the classwork handout on vectors. Solve the following problems.

1. Using vectors, prove that the altitudes of an arbitrary triangle ABC are concurrent (cross at the same point H).
2. Using vectors, prove that the bisectors of an arbitrary triangle ABC are concurrent (cross at the same point O).
3. Using vectors, prove Ceva's theorem.
4. Let $ABCD$ be a square with side a. Point P satisfies the condition, $\overrightarrow{PA} + 3\overrightarrow{PB} + 3\overrightarrow{PC} + \overrightarrow{PD} = 0$. Find the distance between P and the centre of the square, O.
5. Let O and O' be the centroids (medians crossing points) of triangles ABC and $A'B'C'$, respectively. Prove that, $\overrightarrow{OO'} = \frac{1}{3}\left(\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}\right)$.

Algebra: polynomials recap.

Review the classwork notes on polynomials, factorization and Vieta theorem. Solve the following problems; skip those you already solved.

Problems.

1. Write Vieta formulae for the cubic equation, $x^3 + Px^2 + Qx + R = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of P, Q and R,
   a. $(x_1 + x_2 + x_3)^2$
   b. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   c. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   d. $(x_1 + x_2 + x_3)^3$
2. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 6$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$$

$$xy + yz + zx = 11$$

   a. Find a cubic polynomial whose roots are x, y, z
   b. Find x, y, z
3. Find two numbers u, v such that

$$u + v = 6$$

$$uv = 13$$

4. Find three numbers, a, b, c, such that

$$a + b + c = 2$$

$$ab + bc + ca = -7$$

$$abc = -14$$

5. Find all real roots of the following polynomial and factor it.

a. $x^8 + x^4 + 1$
b. $x^4 - x^3 + 5x^2 - x - 6$
c. $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$

6. Perform the long division, finding the quotient and the remainder, on the following polynomials.
a. $(x^3 - 3x^2 + 4) \div (x^2 + 1)$
b. $(x^3 - 3x^2 + 4) \div (x^2 - 1)$