

Complex numbers wrapup.

Exercise. Prove that,

$$\frac{1}{1+x^2} = \frac{1}{2} \left(\frac{1}{1-ix} + \frac{1}{1+ix} \right)$$

Cardano-Tartaglia formula.

Equations involving cubic polynomial are called cubic equations. Roots of a general cubic polynomial are the solutions of an equation,

$$ax^3 + bx^2 + cx + d = 0, a \neq 0 \text{ or}$$

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0, x^3 + Px^2 + Qx + R = 0$$

Using the substitution, $x = y - \frac{b}{3a} = y - \frac{P}{3}$, this can be simplified to a reduced form, which is also called the depressed cubic equation,

$$y^3 + py + q = 0. \text{ Here } p = \frac{3ac-b^2}{3a^2}, q = \frac{2b^3-9abc+27a^2d}{27a^3}.$$

Gerolamo Cardano published a closed formula for the solution of this equation, known as Cardano formula, in his book *Ars Magna* in 1545 (although a closed formula for the roots of a depressed cubic equation was first obtained 6 years earlier by **Nicolo Tartaglia**, who communicated his results to Cardano),

$$y = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

Derivation of Cardano is somewhat esoteric. He uses an heuristic substitution, which was later simplified by Vieta,

$$y = u - \frac{p}{3u}$$

which transforms the original equation, $y^3 + py + q = 0$, to

$$u^3 - \frac{p^3}{27u^3} - up + \frac{p^2}{3u} + p\left(u - \frac{p}{3u}\right) + q = u^3 - \frac{p^3}{27u^3} + q = 0.$$

This is a quadratic equation in $t = u^3$, $t^2 + qt - \frac{p^3}{27} = 0$. Its roots are,

$$t_{1,2} = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}$$

Cardano noticed that this equation does not always have real roots, and therefore a need arises to deal with complex numbers because we know that a cubic equation must have at least one real root. However, he did not know how to deal with this. Nevertheless, if $t_{1,2}$ are real, there is a real u , which is given by a root-three of t ,

$$u = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

While there are two possible choices of \pm sign in the above, they both lead to the same real $y = u - \frac{p}{3u}$, because,

$$\frac{p}{\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = \frac{p \sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3 \left(\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right) \left(\sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right)} = \frac{p \sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3 \left(\sqrt[3]{(-\frac{q}{2})^2 - (\frac{q^2}{4} + \frac{p^3}{27})} \right)} = \frac{p \sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}{3 \left(\sqrt[3]{-\frac{p^3}{27}} \right)},$$

so

$$y = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This is another, equivalent expression for the Cardano formula. Note, that in the case considered by Cardano, where the square root is real, this gives a single real solution, $y = y_0$. However, this solution is obtained by extracting a

root-3 of a real number, and therefore in the field of complex numbers there are three solutions, corresponding to three different roots-3 of unity. If we denote u to be the real root-3, we can write the complex solutions for y as,

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = u\sqrt[3]{1} - \frac{p}{3u\sqrt[3]{1}} = u\sqrt[3]{1} - \frac{p(\sqrt[3]{1})^2}{3u},$$

where the $\sqrt[3]{1}$ has three complex values, thus identifying three complex solutions. Or, equivalently,

$$y = \sqrt[3]{1} \left(\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right) + \frac{1}{\sqrt[3]{1}} \left(\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right).$$

This situation can be exemplified by considering the case $p = 0$, where in addition to the real root, $y_0 = \sqrt[3]{-q}$, there are also two imaginary roots,

$$y_{0,1,2} = y_0 \sqrt[3]{1} = \left\{ y_0, y_0 \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right), y_0 \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) \right\}$$

If, on the other hand, $\frac{q^2}{4} + \frac{p^3}{27} < 0$ and the square root in the discriminant of the quadratic equation for $t = u^3$ is imaginary, then p must be negative, and,

$$\sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = i\sqrt{\left| \frac{q^2}{4} + \frac{p^3}{27} \right|} = i\sqrt{\frac{|p|^3}{27} - \frac{q^2}{4}}, \quad \left| \sqrt[3]{-\frac{q}{2} \pm i\sqrt{\frac{|p|^3}{27} - \frac{q^2}{4}}} \right|^2 =$$

$$\left| \sqrt[3]{\left(\frac{q}{2}\right)^2 + \frac{|p|^3}{27} - \frac{q^2}{4}} \right| = \frac{|p|}{3}.$$

Consequently,

$$\frac{p}{\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = \frac{p \sqrt[3]{-\frac{q}{2} \mp i\sqrt{\frac{|p|^3}{27} - \frac{q^2}{4}}}}{\sqrt[3]{-\frac{q}{2} \pm i\sqrt{\frac{|p|^3}{27} - \frac{q^2}{4}}}} = \frac{p \sqrt[3]{-\frac{q}{2} \mp i\sqrt{\frac{|p|^3}{27} - \frac{q^2}{4}}}}{\sqrt[3]{\frac{|p|}{3}}} = -\sqrt[3]{-\frac{q}{2} \mp i\sqrt{\frac{|p|^3}{27} - \frac{q^2}{4}}},$$

and the equation has three real roots,

$$y = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}} = \sqrt[3]{-\frac{q}{2} + i\sqrt{\frac{p^3}{27} - \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - i\sqrt{\frac{p^3}{27} - \frac{q^2}{4}}},$$

which are given by the three different values of root-3 in the above expression. The roots are real because the expression is a sum of a complex number and its complex conjugate, which is always real. Note, that one has to be careful in selecting which root-3 to use in the above expression. Indeed, $y = u - \frac{p}{3u}$ has only one root-3, that for u , so we have to choose the same value for root-3 in both terms on the right hand side in the Cardano formula above. Both terms are derived from u and they are chosen such that the two above terms are complex conjugate.

Let us consider a special case, $q = 0$. In this case the Cardano formula yields,

$$y_{0,1,2} = \sqrt[3]{\left(\sqrt{\frac{p}{3}}\right)^3} - \frac{p}{\sqrt[3]{\left(\sqrt{\frac{p}{3}}\right)^3}} = \sqrt{\frac{p}{3}} \sqrt[3]{1} - \sqrt{\frac{p}{3}} \frac{1}{\sqrt[3]{1}},$$

where the same choice of root-3 in both terms is required. Or, equivalently,

$$y_{0,1,2} = \sqrt{\frac{p}{3}} \sqrt[3]{1} - \sqrt{\frac{p}{3}} \sqrt[3]{1}, \text{ because } |\sqrt[3]{1}| = 1, \text{ and therefore, } \frac{1}{\sqrt[3]{1}} = \sqrt[3]{1}.$$

In the case $p \geq 0$, we thus obtain, $y_{0,1,2} = \{0, i\sqrt{p}, -i\sqrt{p}\}$. If $p < 0$, the roots are real, $y_{0,1,2} = \{0, \sqrt{|p|}, -\sqrt{|p|}\}$.

Trigonometric substitution for cubic equation.

More consistent derivation of the Cardano formula was given later by Lagrange. Perhaps, the best one is achieved by using a trigonometric substitution, $y = v \cos \theta$, which leads to the equation,

$$v^3 \cos^3 \theta + pv \cos \theta + q = 0.$$

Choosing $v = 2\sqrt{-\frac{p}{3}}$, the equation is reduced to,

$$4 \cos^3 \theta - 3 \cos \theta - \frac{3q}{2p\sqrt{-\frac{p}{3}}} = 0$$

or,

$$\cos(3\theta) = \frac{3q}{2p\sqrt{-\frac{p}{3}}}$$

For more information on cubic equations, see [http://en.wikipedia.org/wiki/Cubic function](http://en.wikipedia.org/wiki/Cubic_function). The only other polynomial equation that is solvable in radicals is the quartic equation, which has been solved by Cardano's student, **Ludovico Ferrari** in 1540. The solution is known as Ferrari formula, and is even more cumbersome than that of Cardano. In fact, it utilizes the latter. It was published by Cardano in his book *Ars Magna* together with the cubic formula in 1545 (<http://en.wikipedia.org/wiki/Quartic equation>).