Homework for April 23, 2023.

Algebra/Geometry. Complex numbers.

Review the classwork handouts on vector and complex numbers. Complete the previous homework assignments. Some problems are repeated below – skip those that you have already solved.

Problems.

1. (i) find the magnitude and the argument, (ii) compute the inverse, and (iii) find the magnitude and the argument of the inverse for the following complex numbers:
   a. $1 + i$
   b. $-i$
   c. $1 + ix$
   d. $\frac{\sqrt{3}}{2} + \frac{i}{2}$
   e. $\frac{1}{2-i} - \frac{1}{2+i}$
2. Find a complex number z whose magnitude is 2 and the argument $Arg(z) = \frac{\pi}{4} = 45°$.
3. Draw the following sets of points on complex plane.
   a. $\{z|Re(z) = 1\}$
   b. $\left\{z|Arg(z) = \frac{3\pi}{4} = 135°\right\}$
   c. $\{z|\ |z| = 1\}$
   d. $\{z|Re(z^2) = 0\}$
   e. $\{z|\ |z^2| = 2\}$
   f. $\{z|\ |z - 1| = 1\}$
   g. $\{z|\ z + \bar{z} = 1\}$
4. Prove that for any complex number z, we have
   a. $|\bar{z}|=|z|$, $Arg(\bar{z}) = -Arg(z)$
   b. $\frac{\bar{z}}{z}$ has magnitude 1; check this for $z = 1 - i$.
5. If z has magnitude 2 and argument $\frac{\pi}{2}$ and w has magnitude 3 and argument $\frac{\pi}{3}$, what will be the magnitude and the argument of zw? Write it in the form $a + bi$.
6. Let $P(x)$ be a polynomial wit real coefficients.
   a. Prove that for any complex number z, we have $\overline{P(z)} = P(\bar{z})$

b. Let z be a complex root of this polynomial, $P(z) = 0$. Prove that then $\bar{z}$ is also a root, $P(\bar{z}) = 0$.

7. Solve the equation $x^3 - 4x^2 + 6x - 4 = 0$. Find the sum and product of the roots in two ways: by using Vieta formulas and by explicit computation. Check that the results match.
8. Simplify following expression:
   a. $(1 + \sin\alpha)(1 - \sin\alpha)$
   b. $(1 + \cos\alpha)(1 - \cos\alpha)$
   c. $\sin^4\alpha - \cos^4\alpha$
9. Prove the following equalities:
   a. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
   b. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$
   c. $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$
   d. $\sin 4\alpha = 4\sin\alpha\cos^3\alpha - 4\cos\alpha\sin^3\alpha$
   e. $\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$
   f. $\cos 5\alpha = \cdots$ (find the expression)
10. Solve the following equation:
    a. $\cos^2\pi x + 4\sin\pi x + 4 = 0$
11. Solve the following equations and inequalities:
    a. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
    b. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$
    c. $\sin^2 x - 2\sin x\cos x = 3\cos^2 x$
    d. $\sin 6x + 2 = 2\cos 4x$
    e. $\cot x - \tan x = \sin x + \cos x$
    f. $\sin x \geq \pi/2$
    g. $\sin x \leq \cos x$
12. Find all complex numbers z such that:
    a. $z^2 = -i$
    b. $z^2 = -2 + 2i\sqrt{3}$
    c. $z^3 = i$

    Hint: write and solve equations for a, b in $z = a + bi$.

13. On the complex plane, plot all fifth order roots of 1 and all fifth order roots of -1.
14.
    a. Find all roots of the polynomial $z + z^2 + z^3 + \cdots + z^n$
    b. Without doing the long division, show that $1 + z + z^2 + \cdots + z^9$ is divisible by $1 + z + z^2 + z^3 + z^4$.

15. Find the roots of the following cubic equations by heuristic guess-and-check factorization, and using the Cardano-Tartaglia formula. Reconcile the two results.

a. $z^3 - 7z + 6 = 0$

b. $z^3 - 21z - 20 = 0$

c. $z^3 - 3z = 0$

d. $z^3 + 3z = 0$

e. $z^3 - \frac{3}{4}z + \frac{1}{4} = 0$

16. Which transformation of the complex plane is defined by:

a. $z \to iz$

b. $z \to \left(\frac{1-i}{\sqrt{2}}\right) z$

c. $z \to (1 + i\sqrt{3})z$

d. $z \to \frac{z}{1+i}$

e. $z \to \frac{z+\bar{z}}{2}$

f. $z \to 1 - 2i + z$

g. $z \to \frac{z}{|z|}$

h. $z \to i\bar{z}$

i. $z \to -\bar{z}$

17. Find the sum of the following trigonometric series using de Moivre formula:

$$S_1 = \cos x + \cos 2x + \cdots + \cos nx = ?$$

$$S_2 = \sin x + \sin 2x + \cdots + \sin nx = ?$$

Geometry. Vectors.

Please, complete problems from the previous homework assignment. Review the classwork handout on vectors. Solve the following problems.

1. Using vectors, prove that the altitudes of an arbitrary triangle ABC are concurrent (cross at the same point H).
2. Using vectors, prove that the bisectors of an arbitrary triangle ABC are concurrent (cross at the same point O).
3. Using vectors, prove Ceva's theorem.
4. Let $ABCD$ be a square with side a. Point P satisfies the condition, $\overrightarrow{PA} + 3\overrightarrow{PB} + 3\overrightarrow{PC} + \overrightarrow{PD} = 0$. Find the distance between P and the centre of the square, O.
5. Let O and O' be the centroids (medians crossing points) of triangles ABC and $A'B'C'$, respectively. Prove that, $\overrightarrow{OO'} = \frac{1}{3}\left(\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}\right)$.

Algebra: polynomials recap.

Review the classwork notes on polynomials, factorization and Vieta theorem. Solve the following problems; skip those you already solved.

Problems.

1. Write Vieta formulae for the cubic equation, $x^3 + Px^2 + Qx + R = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of P, Q and R,
   a. $(x_1 + x_2 + x_3)^2$
   b. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   c. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   d. $(x_1 + x_2 + x_3)^3$
2. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 6$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$$

$$xy + yz + zx = 11$$

a. Find a cubic polynomial whose roots are x, y, z
b. Find x, y, z

3. Find two numbers u, v such that

$$u + v = 6$$

$$uv = 13$$

4. Find three numbers, a, b, c, such that

$$a + b + c = 2$$

$$ab + bc + ca = -7$$

$$abc = -14$$

5. Find all real roots of the following polynomial and factor it.
a. $x^8 + x^4 + 1$
b. $x^4 - x^3 + 5x^2 - x - 6$
c. $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$

6. Perform the long division, finding the quotient and the remainder, on the following polynomials.
a. $(x^3 - 3x^2 + 4) \div (x^2 + 1)$
b. $(x^3 - 3x^2 + 4) \div (x^2 - 1)$