MATH 9
ASSIGNMENT: MAXIMA AND MINIMA
MAY 8, 2022

> For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.
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> Leonhard Euler

First problems on maxima and minima

We considered the following problems in the class:

1. One needs to get from the firefighters station at the point A to the fire at the point B stopping on the way at the bank of the river to get water. At what point at the bank of the river one should stop for water to minimize the total time of travel? Assume that the river is a straight line.
2. The rectangle has a perimeter P. What maximal area could it have? What is the value of that area?
3. Given a triangle ABC find two points P, Q so that the distance PQ is maxtimal. The points can lie both inside and at the boundary of the triangle.

The problem of Dido and Zenodorus Theorem

We talked about the problem of Dido and Zenodorus Theorem.

Theorem 1 (Zenodorus Theorem)**.** *If there exists an n-gon having the maximal area among all n-gons with a given perimeter, it is regular (i.e., the polygon which is both equiangular and equilateral). We will refer to this n-gon as "maximal".*

Lemma 2. *The maximal n-gon is convex.*

Lemma 3. *The maximal n-gon is equilateral.*

Lemma 4. *The maximal n-gon is equiangular.*

We have proven these theorems in the class.

Lemma 5. *The maximal n-gon exists.*

This is not so obvious, e.g., the function $f(x) = -\frac{1}{1+x^2}$ does not reach its maximum. Proof is not given.

Corollary. *The maximal n-gon is a regular n-gon.*

4. Prove that if P and S are the perimeter and the area, respectively, of a given n-gon then
$$P^2 \geq 4n \tan\frac{\pi}{n} S\,.$$
5. Prove that for any polygon
$$4\pi S \leq P^2\,.$$
6. Find the relation between P and S for a circle.

Corollary. *The area of a circle with a given perimeter is greater than the area of any polygon with the same perimeter.*

Lemma 6. *Any curve can be approximated by an n-gon so that the area and the perimeter of the curve are approximated by the ones of the n-gon with any given accuracy.*

Proof is not given.

All of the above leads to the following theorem (the solution of the problem of Dido).

Theorem 7 (Theorem of Dido)**.** *The closed curve of a maximal area for given perimeter is a circle.*

Fermat's principle and Snell's law

We considered the three solutions for the following problem

7. Points A and B are in the upper and lower half planes, respectively. A runner can run with the speed v_1 (v_2) in the upper (lower) half-plane. Find the path from A to B minimizing the total time for the runner.

It turns out that the solution of this problem is the one used by the light propagating in the media with changing refractive index (Snell's law). The statement that light propagates between two points so that it's travel time is minimal is known as Fermat's principle.

Variational calculus

The examples of the problems where the minimization of the function of infinitely many variables is needed are

- Find the shape of the soap film suspended on a given frame (minimizes the area of the surface).
- Find the shape of the chain freely suspended at its ends (minimizes the potential energy of the chain).
- The laws of mechanics can be formulated as follows. The particle of the mass m travels in such a way that the value of the mechanical "action" is minimized. The action is given by the time integral of the difference between the kinetic and the potential energy of the particle.

$$\int_{t_1}^{t_2} dt \left[\frac{m\dot{x}^2}{2} - U(x) \right] .$$

Homework

1. (Kepler plane problem) Find the shape and the area of a rectangle of a maximal area inscribed into the circle of the radius R. (Among all rectangles with the same diagonal, find the rectangle of maximal area).

***2.** Find the shape and the volume of a rectangular parallelepiped inscribed into the sphere of the radius R.

***3.** Find the shape and the volume of a cylinder inscribed into the sphere of the radius R.

4. For a given angle and a point M inside the angle draw a line through M that cuts the triangle of the minimal area of the angle.

5. For a given angle and a point M inside the angle draw a line through M that cuts the triangle of the minimal perimeter of the angle.