Homework 9.

Angular momentum of a rigid body

Last class we discussed conservation of angular momentum. We know how to calculate angular momentum of a point mass which moves along a straight line. Using the calculation rule given in previous homework we can calculate angular momentum of the point mass revolving around point O (see Figure below, right). For the apple revolving around point O the linear velocity is directed along the tangent line, which is perpendicular to the radius drawn to the current position of the apple.

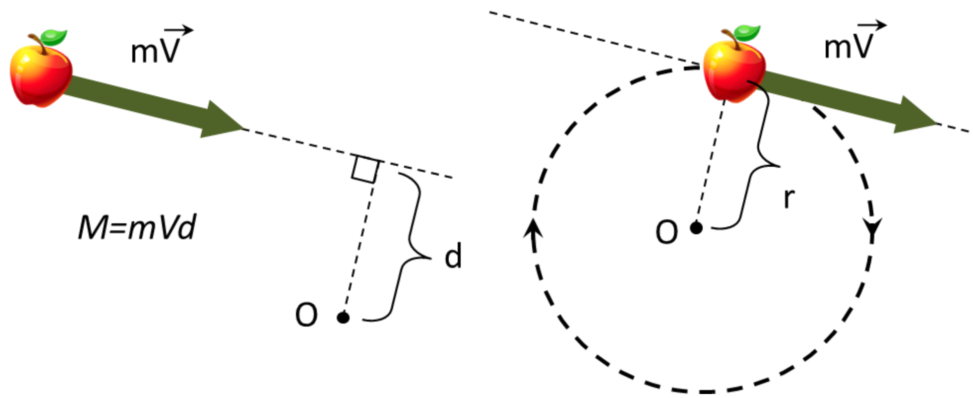

Figure 1.

The *magnitude* of angular momentum **M** for the rotating apple is

$$M = m \cdot V \cdot r \quad (1)$$

As we also remember, angular momentum has both magnitude and direction. Using our right hand rule we can determine that if the apple is revolving clockwise, the angular momentum is directed "from us", perpendicularly to the plane of the picture. In case of rotation apple we can find angular velocity $\boldsymbol{\omega}$:

$$\omega = \frac{V}{d} \Rightarrow V = \omega \cdot d \quad (2)$$

Now we can replace linear velocity in the equation (1) by the product of angular velocity $\boldsymbol{\omega}$ and the rotation radius $\boldsymbol{r}$:

$$M = m \cdot V \cdot r = m \cdot r^2 \cdot \omega = I \cdot \omega \quad (3)$$

In (3) we used the fact that the moment of inertia $\boldsymbol{I}$ of a point mass at a distance $\boldsymbol{r}$ to the rotation axis is:

$$I = m \cdot r^2 \quad (4)$$

Angular velocity has same direction as the angular momentum, so we can write:

$$\vec{M} = I \cdot \vec{\omega} \quad (5)$$

The formula (5) for angular momentum is valid not only for a point mass, but for any rigid body. We can show it same way we calculated the kinetic energy of a rigid body: we separated it (in our mind) to small pieces. Each of the pieces can be approximated by a point mass.

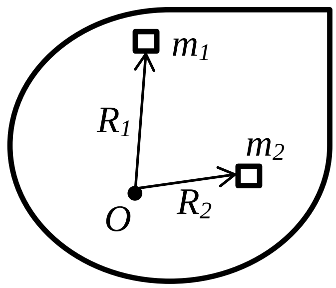

Figure 2.

Again, let us consider a rigid object of arbitrary shape which is rotating with an angular velocity $\boldsymbol{\omega}$ about an axis passing through point *O* (Figure 1) and perpendicular to the picture's plane. Let us "separate" the object to small pieces. The masses of the pieces ($\boldsymbol{m_1}$, $\boldsymbol{m_2}$,…$\boldsymbol{m_i}$) are , generally, not equal to each other. The distances between each piece and the rotation axis (point *O*) are R_1, R_2…R_i. Linear velocity V_1 of piece $\boldsymbol{m_1}$ is:

$$V_1 = R_1 \cdot \omega \qquad (6)$$

Total angular momentum is the sum of angular momenta of all the pieces:

$$\vec{M}_{total} = \vec{M}_1 + \vec{M}_2 + \cdots + \vec{M}_i = \ (7)$$

As long as all the pieces are rotating with the same angular velocity, the direction of all angular momenta is the same and we can write:

$$\vec{M}_{total} = (m_1R_1^2 + m_2R_2^2 + \cdots + m_iR_i^2) \cdot \vec{\omega} = I \cdot \vec{\omega} \ \ (8)$$

Again, in formula (4) the sum $m_1R_1^2 + m_2R_2^2 + \cdots + m_iR_i^2$ is, as we remember, total moment of inertia *I* of the body.

Total angular momentum of an isolated object or system of objects conserves. We have to apply torque to change it. That is why a rolling coin does not flip until it almost stops and that is why the orbit of Earth is so stable.

Problems:

1. What happens to the helicopter if the tail rotor stops working (see picture below).

2. You are sitting on a spinning chair, holding a pair of dumbbells and your hands are stretched to the sides. How will your angular velocity changes if pull your hands in. Let us assume that your initial moment of inertia is decreased two times as you pull your hands in.
3. Return to the problem 2. Calculate how will your kinetic energy change as you pull your hands in. Explain the result.